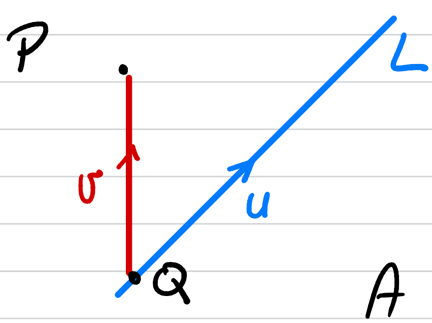
Final - Spring 19 - Solutions

1. Find an equation of the plane that contains the point $(1, 2, -3)$ $\equiv P$ and the line with symmetric equations $x - 2 = y - 1 = \dfrac{z+2}{2}$ $\equiv L$.

A. $5x + y + z = 4$
B. $2x - y + z = -3$
C. $3x + y - 2z = 11$
D. $4x - 2y - 3z = 9$
E. $x + y - 2z = 9$



(i) We identify $\vec{u}$

A normal vector to $x-2 = y-1$ is $\langle 1, -1, 0 \rangle$

A normal vector to $y-1 = \dfrac{z+2}{2}$ is $\langle 0, 2, -1 \rangle$

Thus one can take

$$\vec{u} = \langle 1, -1, 0 \rangle \times \langle 0, 2, -1 \rangle = \langle 1, 1, 2 \rangle$$

(ii) Consider a point on L, e.g for $z=0$.

We get $Q(3, 2, 0)$. Then take

$$\vec{v} = \overrightarrow{PQ} \overset{P(1,2,-3)}{=} \langle 2, 0, 3 \rangle$$

(iii) A normal vector to the plane is

$$\vec{n} = \vec{u} \times \vec{v} = \langle 1, 1, 2 \rangle \times \langle 2, 0, 3 \rangle = \underline{\langle 3, 1, -2 \rangle}$$

This is enough to identify C

2. Identify the surface defined by the equation $x^2 + y^2 + 2z - z^2 = 0$.

A. Ellipse
B. Hyperboloid of one sheet
C. Ellipsoid
D. Hyperboloid of two sheets
E. Paraboloid

First write the equation in normalized way:

$$x^2 + y^2 - (z-1)^2 = -1$$

This surface is thus of the same type as

$$S: \quad x^2 + y^2 - z^2 = -1$$

(see summary quadric surfaces, slides 77-78, chapter on vectors & geometry)

Now S is such that

(i) For $z = z_0^2 \geq 1$, the curve $x^2 + y^2 = z_0^2 - 1$ is an ellipse

(ii) For $y = y_0$, the curve $x^2 - z^2 = -(1 + y_0^2)$ is a hyperbola

Thus S is an hyperboloid of 2 sheets

D

$f \quad g$

3. The vector field $\mathbf{F}(x, y) = \langle \underline{2xe^y + 1}, \underline{x^2 e^y} \rangle$ is conservative. Compute the work done by the field in moving an object along the path $C : \mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$, $0 \le t \le \pi$.

A. -2
B. -1
C. -4
D. -8
E. -6

(i) Since $\vec{F}$ is conservative, we have $\vec{F} = \vec{\nabla}\varphi$. According to Thm 9 in the chapter on vector calculus, we have

$$\int_C \vec{F} \cdot \vec{T}\, ds = \varphi(B) - \varphi(A). \qquad (1)$$

In our case we have $B(-1,0)$ and $A(1,0)$.

(ii) We compute φ according to the recipe on p. 45 (slides on vector calculus).

① $\varphi(x,y) = \int (\overset{f}{\overline{2x e^y + 1}})\, dx = x^2 e^y + x + b(y)$

② $\varphi_y = g \iff x^2 e^y + b'(y) = x^2 e^y$
$\iff b'(y) = 0 \iff b(y) = c$ (constant)

③ We get $\varphi(x,y) = x^2 e^y + x$

(iii) We now apply (1), and we get

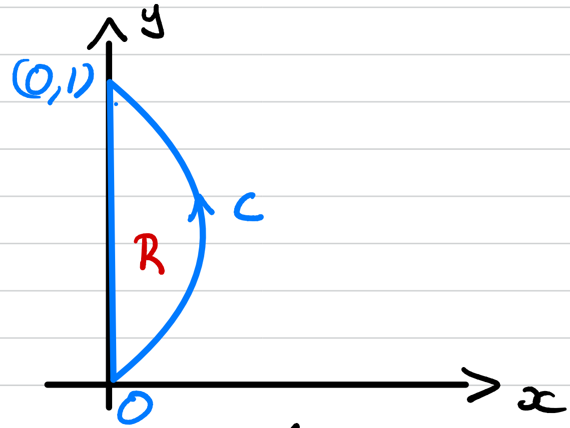
$$\varphi(-1,0) - \varphi(1,0) = (-1)^2 - 1 - (1^2 + 1) = -2$$

Thus $\int_C \vec{F} \cdot \vec{T}\, ds = -2$ Ⓐ

4. Compute

$$\int_C \overset{f}{\underline{(e^{2x}+y^2)}}\,dx + \overset{g}{\underline{(14xy+y^2)}}dy, \qquad \vec{F} = \langle f, g \rangle$$

where C is the boundary of the region bounded by the y-axis and the curve $x = y - y^2$ oriented counterclockwise.



A. 1
B. 2
C. 4
D. 12
E. 24

(i) In order to avoid a parametrization of C, we will use Green's theorem (Thm 13, slides on vector calculus), and evaluate $\int_C \vec{F}\cdot d\vec{r}$ as

$$\iint_R \mathrm{Curl}(\vec{F})\, dA \qquad (1)$$

(ii) In (1), we have (see Def 10, vector calculus)

$$\mathrm{Curl}(\vec{F}) = g_x - f_y = 14y - 2y = 12y$$

We also have $R = \{ 0 \le y \le 1,\ 0 \le x \le y - y^2 \}$

(iii) The integral in (1) is computed as

$$\iint_R \mathrm{Curl}(\vec{F})\, dA = \int_0^1 \int_0^{y-y^2} 12y \, dx\, dy$$

$$= 12 \int_0^1 y(y-y^2)\, dy = 12 \int_0^1 (y^2 - y^3)\, dy$$

$$= 12\left(\frac{1}{3} - \frac{1}{4}\right) = 1$$

A

5. Find the linear approximation of $f(x, y) = y\sqrt{x}$ at $(4, 1)$.

A. $\frac{1}{4}x + 16y - 15$

B. $\frac{1}{4}x + 8y - 7$

C. $\frac{1}{4}x + 4y - 3$

D. $\frac{1}{4}x + y + 1$

E. $\frac{1}{4}x + 2y - 1$

(i) The linear approximation of a 2-d function f is given in Def 11, slides on Functions of several variables. It can be read as

$$f(x,y) \simeq f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

(ii) Application: $f(x,y) = y\, x^{1/2}$, $a = 4$, $b = 1$.

Then $f(a,b) = 2$

$f_x(x,y) = \frac{1}{2} y\, x^{-1/2}$ $\qquad f_x(a,b) = \frac{1}{4}$

$f_y(x,y) = x^{1/2}$ $\qquad f_y(a,b) = 2$

(iii) The linear approximation is

$$f(x,y) \simeq 2 + \frac{1}{4}(x-4) + 2(y-1)$$

$$f(x,y) \simeq \frac{1}{4}x + 2y - 1$$

E

6. Compute $\operatorname{curl}\mathbf{F}(\pi, 1, 1)$, where $\mathbf{F} = \langle \overset{f}{x+y}, \overset{g}{yz}, \overset{h}{\sin(x)} \rangle$.

A. $\langle 1, 1, -1 \rangle$
B. $\langle 1, 1, 1 \rangle$
C. $\langle -1, 1, -1 \rangle$
D. $\langle -1, -1, -1 \rangle$
E. $\langle 1, -1, -1 \rangle$

The definition of curl is taken from Def 17, slides on Vector Calculus. We get

$$\operatorname{Curl}(\vec{F}) = \left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \partial_x & \partial_y & \partial_z & \partial_x & \partial_y \\ x+y & yz & \sin(x) & x+y & yz \end{array}\right.$$

$$= \vec{i}\,(0 - y) + \vec{j}\,(0 - \cos(x)) + \vec{k}\,(0 - 1)$$

$$= -\langle y, \cos(x), 1 \rangle$$

For $\langle x, y, z \rangle = \langle \pi, 1, 1 \rangle$ we get

$$\underline{\operatorname{Curl}(\vec{F})} = -\langle 1, -1, 1 \rangle = \underline{\langle -1, 1, -1 \rangle}$$

7. If $f(x, y) = x \sin(xy^2)$, compute $f_{yx}(\pi, 1)$.

A. -8π
B. -6π
C. -2π
D. $-\pi$
E. -4π

We have

$$f_y = x \times 2xy \cos(xy^2)$$
$$= 2y x^2 \cos(y^2 x)$$

$$f_{yx} = 2y\left(2x \cos(y^2 x) - x^2 y^2 \sin(y^2 x)\right)$$
$$= 2xy\left(2 \cos(y^2 x) - x y^2 \sin(y^2 x)\right)$$

If $x = \pi$, $y = 1$ we get

$$f_{yx}(\pi, 1) = 2\pi\left(2 \underbrace{\cos(\pi)}_{=-1} - \pi \underbrace{\sin(\pi)}_{=0}\right)$$

$$f_{yx}(\pi, 1) = -4\pi$$

E

8. Find the direction in which $f(x, y, z) = \frac{x}{y} - yz$ decreases most rapidly at the point $(4, 1, 1)$?

A. $\frac{1}{\sqrt{27}}\langle 1, -5, 1\rangle$

B. $\frac{1}{\sqrt{27}}\langle 1, -5, -1\rangle$

C. $\frac{1}{\sqrt{27}}\langle -1, 5, -1\rangle$

D. $\frac{1}{\sqrt{27}}\langle -1, 5, 1\rangle$

E. $\frac{1}{\sqrt{27}}\langle 1, 5, 1\rangle$

(i) According to slides on Functions of Several Variables, p. 63, the direction of maximal descent is given by

$$\vec{u} = -\frac{\vec{\nabla} f(x,y)}{|\vec{\nabla} f(x,y)|} \qquad (1)$$

(ii) We compute $\vec{\nabla} f$, for $f(x,y,z) = \frac{x}{y} - yz$

$$\vec{\nabla} f(x,y,z) = \left\langle \frac{1}{y}, -\frac{x}{y^2} - z, -y \right\rangle$$

Thus $\vec{\nabla} f(4,1,1) = \langle 1, -5, -1 \rangle$

$$|\vec{\nabla} f(4,1,1)| = (1^2 + 5^2 + 1^2)^{\frac{1}{2}} = \sqrt{27}$$

(iii) According to (1) we get

$$\vec{u} = -\frac{1}{\sqrt{27}} \langle 1, -5, -1 \rangle$$

$$\vec{u} = \frac{1}{\sqrt{27}} \langle -1, 5, 1 \rangle$$

D

9. Let M and m denote the maximum and the minimum values of $f(x, y) = x^2 - 2x + y^2 + 3$ in the disk $x^2 + y^2 \leq 1$. Find $M + m$.

Region $\mathcal{R}$

We follow the recipe given by Prop 14 in the slides on Functions of Several Variables.

A. 4
B. 5
C. 12
D. 8
E. 7

① We have

$f_x = 2x - 2 \qquad f_y = 2y$

Thus $\vec{\nabla} f = 0 \iff \begin{cases} 2x - 2 = 0 \\ 2y = 0 \end{cases}$

$\iff (x, y) = (1, 0)$ critical point in $\mathcal{R}$

$f(1,0) = 2$

② The boundary of $\mathcal{R}$ is parametrized as

$\{ (\cos(t), \sin(t)) ; \quad 0 \leq t \leq 2\pi \}$

Then

$f(\cos(t), \sin(t)) = \cos^2(t) - 2\cos(t) + \sin^2(t) + 3$

$= 4 - 2\cos(t) \equiv g(t)$

We get:

max: for $t = \pi$, $\quad g(\pi) = 6$

min: for $t = 0$, $\quad g(0) = 2$

③ Putting together the values we have found in ① & ②, we get

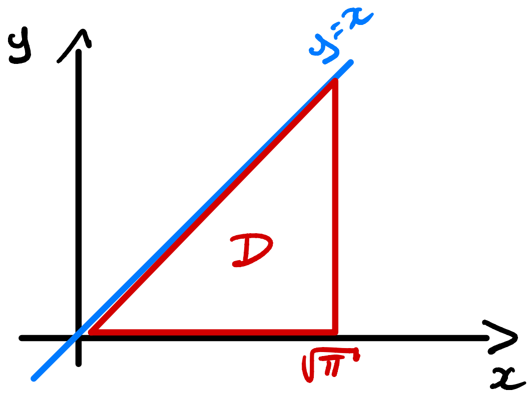
$m = 2, \quad M = 6 \qquad \underline{M + m = 8}$

D

$\equiv I$

10. Evaluate the integral $\iint_D 2\pi \sin(x^2)\, dA$ where D is the region in the xy-plane bounded by the lines $y = 0$, $y = x$ and $x = \sqrt{\pi}$.

A. 2π
B. π
C. 4π
D. 8π
E. $\pi/2$



Write $D = \{ 0 \le x \le \sqrt{\pi},\ 0 \le y \le x \}$

Then

$$I = \int_0^{\sqrt{\pi}} \int_0^{x} 2\pi \sin(x^2)\, dy\, dx$$

$$= \pi \int_0^{\sqrt{\pi}} 2x \sin(x^2)\, dx$$ → of the form $u' \sin(u)$

$$= \pi \left. -\cos(x^2) \right|_0^{\sqrt{\pi}}$$

$$I = 2\pi$$

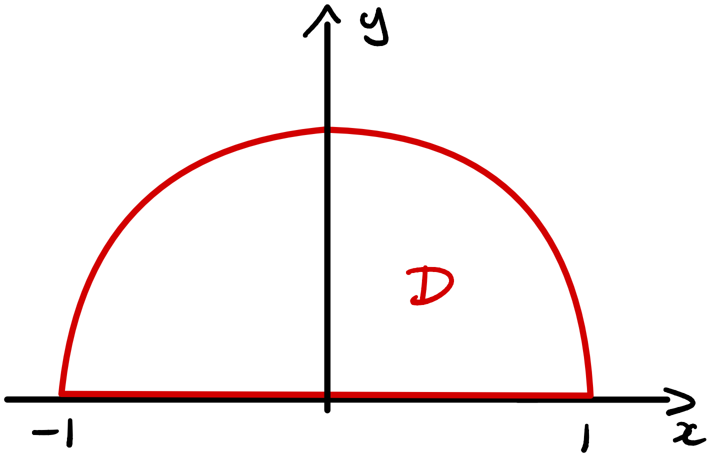
A

11. Evaluate the double integral

$$I \equiv \iint_D \overbrace{2e^{(x^2+y^2)}}^{f(x,y)}\, \mathrm{dA},$$

where D is the region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$.

A. $8\pi(e-1)$
B. $2\pi(e-1)$
C. $4\pi(e-1)$
D. $\pi(e-1)$
E. $16\pi(e-1)$



Since f is a function of x^2+y^2 and D is part of a ball, we use polar coordinates. We get

$$D = \{(r,\theta);\ 0 \le r \le 1,\quad 0 \le \theta \le \pi\}$$

Thanks to Thm 2 in the slides on Multiple Integration we get

$$I = \int_0^\pi \int_0^1 \underbrace{2e^{r^2} r}_{\text{of the form } u'e^u}\, dr\, d\theta$$

$$= \int_0^\pi e^{r^2}\Big|_0^1\, d\theta = (e^1-1)\int_0^\pi d\theta$$

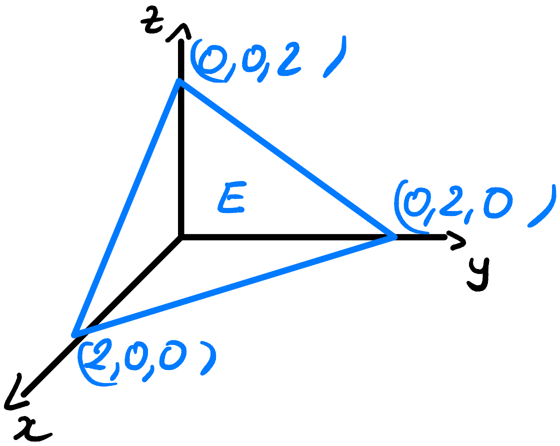
Thus $\underline{I = \pi(e-1)}$ D

12. Compute the triple integral

$$I = \iiint_E 3y\,\mathrm{dV},$$

where E is a region under the plane $x + y + z = 2$ in the first octant.

A. 4
B. 2
C. 6
D. 3
E. 1



The domain E can be expressed as

$$E = \{ 0 \le x \le 2,\ 0 \le y \le 2-x,\ 0 \le z \le 2-x-y \}$$

Thus we have

$$I = 3\int_0^2 \int_0^{2-x} \int_0^{2-x-y} y \, dz\, dy\, dx$$

$$= 3\int_0^2 \int_0^{2-x} \left((2-x)y - y^2\right) dy\, dx$$

$$= 3\int_0^2 \left((2-x) \times \frac{(2-x)^2}{2} - \frac{1}{3}(2-x)^3\right) dx$$

$$= \frac{3}{6}\int_0^2 (2-x)^3\, dx \overset{u := 2-x}{=} \frac{1}{2}\int_0^2 u^3\, du$$

$$= \frac{1}{8} u^4 \Big|_0^2 = 2$$

We get $\underline{I = 2}$ B

13. The integral

$$I \equiv \int_0^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{8-x^2-y^2}} xy^2 z \, dz \, dy \, dx$$

when converted to cylindrical coordinates becomes

A. $\int_{-\pi/2}^{\pi/2} \int_0^2 \int_{\sqrt{3}r}^{\sqrt{8-r^2}} r^4 z \cos\theta \sin^2\theta \, dz \, dr \, d\theta$

B. $\int_{-\pi/2}^{\pi/2} \int_0^{\sqrt{2}} \int_{\sqrt{3}r}^{\sqrt{8-r^2}} r^3 z \cos\theta \sin^2\theta \, dz \, dr \, d\theta$

C. $\int_{-\pi/2}^{\pi/2} \int_0^{\sqrt{2}} \int_{\sqrt{3}r}^{\sqrt{8-r^2}} r^4 z \cos\theta \sin^2\theta \, dz \, dr \, d\theta$

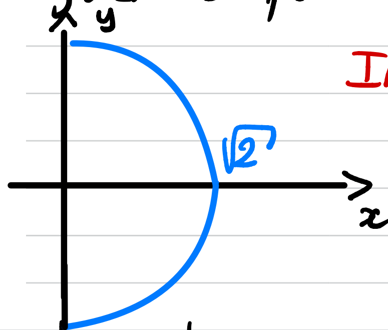
D. $\int_0^{\pi} \int_0^{\sqrt{2}} \int_{\sqrt{3}r}^{\sqrt{8-r^2}} r^4 z \cos\theta \sin^2\theta \, dz \, dr \, d\theta$

E. $\int_0^{\pi} \int_0^{2} \int_{\sqrt{3}r}^{\sqrt{8-r^2}} r^4 z \cos\theta \sin^2\theta \, dz \, dr \, d\theta$

In the slides on Multiple Integration, one can find

(a) Conversion Cartesian / cylindrical on p. 62

(b) Integration formula on p. 68



In xy-plane

Here the domain of integration can be expressed as

$$\{ 0 \le r \le \sqrt{2}, \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \ \sqrt{3}\, r \le z \le (8-r^2)^{\frac{1}{2}} \}$$

Then

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} \int_{\sqrt{3}r}^{\sqrt{8-r^2}} \underbrace{r\cos\theta}_{x} \ \underbrace{r^2 \sin^2\theta}_{y^2} \ z \ r \, dz \, dr \, d\theta$$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} \int_{\sqrt{3}r}^{(8-r^2)^{\frac{1}{2}}} r^4 z \cos\theta \sin^2\theta \, dz \, dr \, d\theta$$

C

14. Convert the integral to spherical coordinates and compute it:

$$I = \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} 3\,dz\,dy\,dx.$$

A. $2(\sqrt{2}-1)\pi$
B. $8(\sqrt{2}-1)\pi$
C. $10(\sqrt{2}-1)\pi$
D. $16(\sqrt{2}-1)\pi$
E. $12(\sqrt{2}-1)\pi$

In slides on Multiple Integration we have

(a) Conversion Spherical / Cartesian on p. 73

(b) Integration formula on p. 78

Here we have a volume delimited by

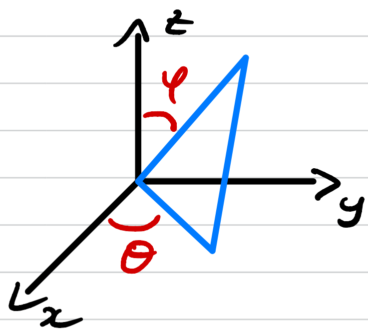
(i) $z^2 = x^2+y^2$: cone $\varphi = \frac{\pi}{4}$

(ii) $z^2 = 8 - x^2 - y^2$: sphere $\rho = \sqrt{8}$

Note that since $\varphi \le \frac{\pi}{4}$, we have

$x^2+y^2 = \rho^2 \sin^2\varphi \le 4$,

which is compatible with the bounds of integration. Because $0 \le y \le \sqrt{4-x^2}$, we have $0 \le \theta \le \pi$. Our domain of integration is



$$\left\{ 0 \le \rho \le \sqrt{8},\quad 0 \le \varphi \le \frac{\pi}{4},\quad 0 \le \theta \le \pi \right\}.$$

Computation of I

$$I = 3\int_0^{\pi} \int_0^{\pi/4} \int_0^{\sqrt{8}} \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= 3\pi \int_0^{\pi/4} \sin(\varphi)\, d\varphi \int_0^{\sqrt{8}} \rho^2\, d\rho$$

$$= 3\pi \left(-\cos(\varphi)\right)\Big|_0^{\frac{\pi}{4}} \quad \frac{1}{3}\left(2\sqrt{2}\right)^3$$

$$= \pi \left(1 - \frac{1}{\sqrt{2}}\right) \times 8 \times 2\sqrt{2}$$

$$I = 16\pi\left(\sqrt{2} - 1\right)$$

D

15. Compute the line integral

$$I \equiv \int_C \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F} = \langle xy, x+y \rangle$ and C is the curve $y = x^2$ from $(0,0)$ to $(1,1)$.

A. $\frac{13}{12}$

B. $\frac{21}{12}$

C. $\frac{17}{12}$

D. $\frac{5}{12}$

E. $\frac{23}{12}$

Note that $\vec{F}$ is not conservative. Thus we simply evaluate I invoking Thm 5 in the slides on Vector Calculus.

A parametrization of C is

$$C : \{ \vec{r}(t) = \langle t, t^2 \rangle ; \quad 0 \le t \le 1 \}$$

Thus $\vec{r}\,'(t) = \langle 1, 2t \rangle$ and

$$I = \int_0^1 \langle \underbrace{t \times t^2}_{xy}, \underbrace{t + t^2}_{x+y} \rangle \cdot \langle 1, 2t \rangle \, dt$$

$$= \int_0^1 \left(t^3 + 2t\,(t + t^2) \right) dt$$

$$= \int_0^1 (3t^3 + 2t^2)\, dt$$

$$= \frac{3}{4} + \frac{2}{3}$$

$$I = \frac{17}{12}$$

C

16. Let S be the part of the surface $z = \underbrace{xy + 1}_{\equiv g(x,y)}$ that lies within the cylinder $x^2 + y^2 = 1$. Find the area of the surface S.

A. $\frac{\sqrt{2}}{3}\pi - \frac{2}{3}\pi$

B. $\frac{\sqrt{2}}{3}\pi - \frac{1}{3}\pi$

C. $\frac{4\sqrt{2}}{3}\pi - \frac{1}{3}\pi$

D. $\frac{4\sqrt{2}}{3}\pi - \frac{2}{3}\pi$

E. $\frac{2\sqrt{2}}{3}\pi - \frac{2}{3}\pi$

The surface S is given explicitly by $z = xy+1$. Therefore one can use Thm 20 in the slides on Vector Calculus. Note that $z_x = y$, $z_y = x$.

We get $\text{Area}(S) = \iint_{x^2+y^2 \le 1} (z_x^2 + z_y^2 + 1)^{\frac{1}{2}}\, dx\, dy$

$\text{Area}(S) = \iint_{x^2+y^2 \le 1} (x^2 + y^2 + 1)^{\frac{1}{2}}\, dx\, dy$

A polar coordinate change of variables seems to be in order. We obtain

$$\text{Area}(S) = \frac{1}{2}\int_0^{2\pi}\int_0^1 \underbrace{(r^2+1)^{\frac{1}{2}} 2r}_{\text{of the form } u'(u+1)^{\frac{1}{2}}}\, dr\, d\theta$$

$$= \frac{1}{2} \times \frac{2}{3} \int_0^{2\pi} (r^2+1)^{\frac{3}{2}} \Big|_0^1$$

$$= \frac{2\pi}{3}\left(2^{3/2} - 1\right)$$

$$\text{Area}(S) = \frac{4\pi}{3}\sqrt{2} - \frac{2\pi}{3}$$

D

$= S$

17. Find the surface area of the parametric surface $\mathbf{r}(u,v) = \langle u^2, uv, v^2/2\rangle$ with $0 \le u \le 3$, $0 \le v \le 1$.

A. 12
B. 15
C. 18
D. 19
E. 27

The surface is given with a parametric description. Thus we use Thm 19 in the slides on Vector Calculus. We need to compute first

$$\vec{t}_u = \langle 2u, v, 0\rangle \qquad \vec{t}_v = \langle 0, u, v\rangle$$

$$\vec{t}_u \times \vec{t}_v = \langle v^2, -2uv, 2u^2\rangle$$

$$|\vec{t}_u \times \vec{t}_v| = (v^4 + 4u^2v^2 + 4u^4)^{\frac{1}{2}}$$
$$= ((2u^2+v^2)^2)^{\frac{1}{2}} = 2u^2+v^2$$

Thus

$$\text{Area}(S) = \int_0^3 \int_0^1 (2u^2+v^2)\, dv\, du$$

$$= \int_0^3 2u^2v + \frac{1}{3}v^3 \Big|_0^1 \, du$$

$$= \int_0^3 \left(2u^2 + \frac{1}{3}\right) du$$

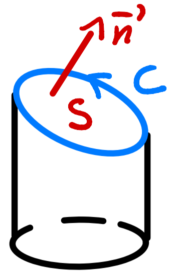
$$= \frac{2}{3}u^3\Big|_0^3 + 3\times\frac{1}{3} = 2\times 9 + 1$$

$$\text{Area}(S) = 19$$

D

18. Use Stokes' Theorem to evaluate the integral $\underbrace{\int_C y\,dx + z\,dy + x\,dz}_{\equiv I}$, where C is the intersection of the surfaces $x^2+y^2=1$ and $x+y+z=5$. C is oriented counterclockwise when viewed from above.

A. -8π
B. -6π
C. $-\pi$
D. -3π
E. -9π



Stokes' theorem is Thm 23 in the slides on Vector Calculus. The surface S will be chosen as the lid delimited by C. We get

$$S : \{ \vec{r}(u,v) = \langle u, v, 5-u-v \rangle ;\ u^2+v^2 \le 1 \}$$

The normal to the plane/lid is $\langle 1,1,1\rangle = \vec{n}$

Moreover, $\vec{F} = \langle y, z, x \rangle$

$$\mathrm{Curl}(\vec{F}) = -\langle 1,1,1\rangle$$

Hence

$$I = \iint_{(u^2+v^2\le 1)} \underbrace{-\langle 1,1,1\rangle}_{\mathrm{curl}(\vec{F})} \cdot \underbrace{\langle 1,1,1\rangle}_{\vec{n}}\, du\, dv$$

$$= -3\ \mathrm{Area}(\text{circle } x^2+y^2 \le 1)$$

$$\underline{I = -3\pi}$$

D

$=I$

19. Evaluate the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle 3xy^2, x\cos(z), z^3 \rangle$ and S is the complete boundary surface of the solid region bounded by the cylinder $y^2 + z^2 = 2$ and the planes $x = 1$ and $x = 3$. S is oriented by the outward normal.

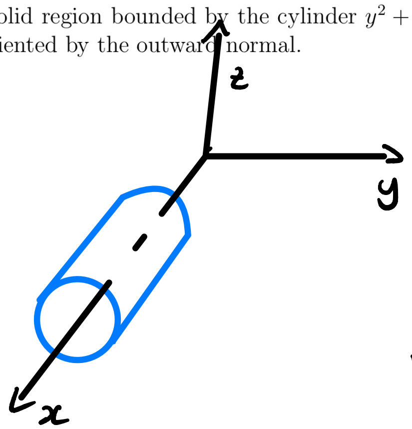
A. 9π

B. 12π

C. 14π

D. 18π

E. 24π



We use the divergence Thm 24 in the slides on Vector calculus.

We have $D = \{ 1 \le x \le 3,\ y^2+z^2 \le 2 \}$

If we use cylindrical coordinates $\langle x, r\cos(\theta), r\sin(\theta) \rangle$

we get $D = \{ 1 \le x \le 3,\ 0 \le r \le \sqrt{2},\ 0 \le \theta \le 2\pi \}$

We also have $\vec{F} = \langle 3xy^2, x\cos(z), z^3 \rangle$ and thus

$$\mathrm{Div}(\vec{F}) = 3y^2 + 3z^2 = 3r^2$$

Therefore

$$I = \int_1^3 \int_0^{2\pi} \int_0^{\sqrt{2}} 3r^2 \times r \, dr \, d\theta \, dx$$

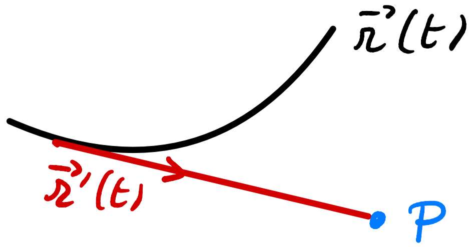
$$= 2 \times 2\pi \times \frac{3}{4} r^4 \Big|_0^{\sqrt{2}} = 3 \times \sqrt{2}^4 \times \pi$$

$\underline{I = 12\pi}$

B

P

20. The position function of a Space Shuttle is $\mathbf{r}(t) = \langle t^2, -t, 6\rangle$, $t \geq 0$. The International Space Station has coordinates $\boxed{(16, -5, 6)}$. In order to dock the Space Shuttle with the Space Station the captain plans to turn off the engine so that the Space Shuttle coasts into the Space Station. At what time should the captain turn off the engines? Assume there are no other forces acting on the Space Shuttle other than the force of the engine.



A. 6
B. 8
C. 2
D. 4
E. 0

When the engine is stopped, the velocity becomes constant. Hence we can rephrase the problem as: when does the line following the tangent to $\vec{r}$ hit P?

We have $\vec{r}\,'(t) = \langle 2t, -1, 0\rangle$. Thus according to Prop 8 (slides on Vectors & Geometry) the equation of the line at time t is

$$\langle t^2, -t, 6\rangle + u\langle 2t, -1, 0\rangle$$

$$= \langle t^2 + 2tu, -t-u, 6\rangle, \quad u \geq 0$$

same coordinate as P

Hence we want to find $u, t \geq 0$ such that

$$t^2 + 2tu = 16, \quad t + u = 5 \qquad (1)$$

We substitute $t = 5-u$ in the 1st equation. We get

$$t^2 + 2t(5-t) - 16 = 0$$

$$\Leftrightarrow \quad t^2 - 10t + 16 = 0 \qquad \text{roots: } t = 2, 8$$

Moreover, $t = 8$ would give $u = -3$ in (1).

Hence only $t = 2$ works

C