Final - Fall 17 - Solutions

1. A line L contains the point $(1, 4, -3)$ and is parallel to to the line

$$x = 5 + 3t, \quad y = 1 - t, \quad z = 1 + 3t.$$

What point on L intersects the plane $y = 0$?

Equation for the line

The direction is $\vec{v} = \langle 3, -1, 3 \rangle$

we also have $P_0(1, 4, -3)$

Thus the line is given by

$$\langle x, y, z \rangle = \langle 1 + 3t, 4 - t, -3 + 3t \rangle$$

Intersection with the plane

we have

$$y = 0 \iff t = 4$$

The corresponding point is

$$\langle 1 + 3 \times 4, 0, -3 + 3 \times 4 \rangle$$

$$\boxed{\langle 13, 0, 9 \rangle}$$

E

2. The plane passing through the point $(0, 1, -1)$ and parallel to the plane $x + y + 2z = 3$ intersects the x-axis at the point:

The plane is such that

(i) Normal $\equiv \vec{n} = \langle 1, 1, 2 \rangle$

(ii) Passes through $P_0(0, 1, -1)$

Equation

$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$

$$\iff \langle 1, 1, 2 \rangle \cdot \langle x, y-1, z+1 \rangle = 0$$

$$\iff x + y + 2z = -1$$

Intersection on the x-axis,

$y = 0$, $z = 0$. Hence

$$x = -1$$

The intersection is

$$\boxed{(-1, 0, 0)}$$

A

3. The position of a particle is given by $\mathbf{r}(t) = \langle 2t,\, 1-2t,\, 5+t\rangle$, starting when $t = 0$. After the particle has gone a *distance* of 3, the x-coordinate is

Write

$$\vec{r}(t) = \langle 0, 1, 5\rangle + t\,\overbrace{\langle 2, -2, 1\rangle}^{\equiv \vec{v}}$$

Next

$$|\vec{v}| = (4+4+1)^{\frac{1}{2}} = 3$$

Hence

$\vec{r}(t)$ has gone a distance of 3

$\Leftrightarrow \quad t = 1$

For $t = 1$

$$\vec{r}(t) = \langle 2, -1, 6\rangle$$

In particular,

$$\boxed{x(t) = 2}$$

D

$f(x,y,z) = x^2+4xy+z^4+11$

4. The tangent plane to the level surface $x^2 + 4xy + z^4 = -11$ at the point $(2, -2, 1)$ is given by the equation

Namal to tangent plane Given by

$\vec{n} = \nabla f = \langle 2x+4y, 4x, 4z^3 \rangle$

At $(2,-2,1)$ we get

$\vec{n} = \langle -4, 8, 4 \rangle$

Simplified $\vec{n}$ one can take

$\vec{n} = \langle -1, 2, 1 \rangle$

Equation fu the plane

$-x+2y+z = -2+2\times(-2)+1$

$\Leftrightarrow -x+2y+z = -5$

$\Leftrightarrow \boxed{x-2y-z = 5}$

C

5. Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\frac{2xy}{x^4+4y^2}$$

Which of the following statements are true?

(i) The function is continuous at $(0,0)$.

(ii) $\dfrac{\partial f}{\partial x}(0,0) = 0$.

(iii) The function is differentiable outside of $(0,0)$.

(i) $f(x,x) = \frac{1}{2}$ and $f(x,2x) = \frac{2}{5}$

Hence

$$\lim_{y=x,\, x\to 0} f(x,y) \neq \lim_{y=2x,\, x\to 0} f(x,y)$$

f is not continuous at $(0,0)$

(ii) $f(x,0) - f(0,0) = 0$ for all $x \neq 0$. Hence

$\partial_x f(0,0) = 0$

(iii) f is a ratio of polynomials

$\Rightarrow$ f differentiable whenever $x^2+y^2 \neq 0$

$\Rightarrow$ f differentiable outside of $(0,0)$

(ii) and (iii) true B

6. Suppose $f(x, y) = g(x)h(y)$, where g and h are continuously differentiable functions of one variable with $g(1) = 2$, $g'(1) = 3$, $h(2) = 5$, and $h'(2) = -1$. Approximate $f(1.1, 2.2)$.

Formula For $(a,b) = (1,2)$ we have

$f(x,y)$ (1)

$\simeq f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Derivatives Since $f(x,y) = g(x)h(y)$,

- $f_x = g'(x)h(y)$
- $f(1,2) = 10$

$\Rightarrow f_x(1,2) = g'(1)h(2) = 15$

- $f_y = g(x)h'(y)$

$\Rightarrow f_y(1,2) = g(1)h'(2) = -2$

Approximation Plugging the values in (1),

$f(1.1, 2.2) \simeq 10 + 15 \times 0.1 - 2 \times 0.2$

$\boxed{f(1.1, 2.2) \simeq 11.1}$

E

$2t = \frac{3\pi}{2}$

$2t = \frac{5\pi}{2}$

7. The number and value of the absolute maxima of the function $f(x, y) = x^2 - xy + y^2$ on the domain $2x^2 + 2y^2 \leq 1$ is $\to R$

we use a two steps procedure

(i) Critical points

$$\nabla f(x,y) = \langle 2x - y \, , \, 2y - x \rangle$$

Hence $\nabla f(x,y) = \langle 0,0 \rangle$ point in R

$$\Longleftrightarrow \begin{cases} 2x - y = 0 \\ -x + 2y = 0 \end{cases} \Longleftrightarrow (x,y) = (0,0)$$

and $\boxed{f(0,0) = 0}$

(ii) Boundary The boundary of R is the circle $C: x^2 + y^2 = \frac{1}{2}$.

It is parametrized as

$$C: \left\{ \left(\frac{1}{\sqrt{2}} \cos t , \frac{1}{\sqrt{2}} \sin t \right) ; 0 \leq t \leq 2\pi \right\},$$

and

$$f\left(2^{-1/2} \cos t , 2^{-1/2} \sin t \right)$$

$$= \frac{1}{2} - \frac{1}{2} \sin t \cos t$$

$$= \frac{1}{2} - \frac{1}{4} \sin(2t)$$

Max for

$$\boxed{\left. \begin{array}{l} t = \frac{3\pi}{4} \\ t = \frac{5\pi}{4} \end{array} \right\} f = \frac{3}{4}}$$

(iii) Conclusion

we have $f_{max} = \frac{3}{4}$, obtained at 2 points on the boundary

Ⓔ

$$\overbrace{\iint_R \cos(x+y)\,dA}^{=I}$$

8. Compute the double integral $\iint_R \cos(x+y)\,dA$, where R is the rectangle $[0,\pi]\times[0,\pi]$.

We have

$$I = \int_0^\pi \int_0^\pi \cos(x+y)\,dy\,dx$$

$$= \int_0^\pi \sin(x+y)\Big|_0^\pi \,dx$$

$$= \int_0^\pi \left(\sin(x+\pi) - \sin(x)\right) dx$$

$$= -2\int_0^\pi \sin(x)\,dx$$

$$= 2\cos(x)\Big|_0^\pi$$

$$\boxed{I = -4}$$

A

9. Find $\overbrace{\iiint_E yz\, dV}^{I}$ where E is the solid tetrahedron bounded by the planes $z = 0$, $y = z$, $y = x$, and $x = 1$.

The domain can be described as

$$E = \{0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le y\}$$

Hence

$$I = \int_0^1 \int_0^x \int_0^y yz\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^x y \times \frac{y^2}{2}\, dy\, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^x y^3\, dy\, dx$$

$$= \frac{1}{8} \int_0^1 x^4\, dx$$

$$\boxed{I = \frac{1}{40}}$$

B

10. The density of a solid sphere at any point is proportional to the point's distance from the center of the sphere. What is the ratio of the mass of a sphere of radius 1 to a sphere of radius 2?

proportional constant

The density f is $f(x,y,z) = a\rho$, where $\rho =$ distance to origin. Call B_1 the sphere with radius 1, B_2 the sphere with radius 2. Then in spherical coordinates

$$m_1 = \iiint_{B_1} f \, dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 a\rho \times \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= 2\pi a \int_0^{\pi} \sin\varphi \, d\varphi \times \int_0^1 \rho^3 \, d\rho$$

$$= 4\pi a \times \frac{1}{4}$$

For B_2, the same computation gives

$$m_2 = 4\pi a \times \int_0^2 \rho^3 \, d\rho = 4\pi a \times 4$$

The ratio is

$$\boxed{\frac{m_1}{m_2} = \frac{1}{16}}$$

D

S

11. Find the area of the portion of the plane $x + 2y + 2z = 2$ that lies in the first octant.

Strategy Write $2z = 2 - x - 2y$

$$\Leftrightarrow z = 1 - \frac{x}{2} - y$$

Then one can use the surface area formula for explicit functions.

Domain If $x, y, z \geq 0$, the domain of integration is

$$R = \{ 0 \leq x \leq 2 , \quad 0 \leq y \leq 1 - \frac{x}{2} \}$$

Surface We compute

$$S = \int_0^2 \int_0^{1-\frac{x}{2}} (1 + z_x^2 + z_y^2)^{1/2} \, dy \, dx$$

$$= \int_0^2 \int_0^{1-x/2} \left(1 + \frac{1}{4} + 1\right)^{1/2} dy \, dx$$

$$= \frac{3}{2} \int_0^2 \left(1 - \frac{x}{2}\right) dx$$

$$= \frac{3}{2} \left(2 - \frac{x^2}{4} \Big|_0^2 \right)$$

$$\boxed{S = \frac{3}{2}}$$

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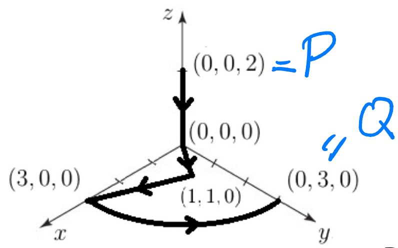
12. The oriented curve C consists of the line segment from $(0,0,2)$ to $(0,0,0)$, followed by the line segment from $(0,0,0)$ to $(1,1,0)$, followed by the line segment from $(1,1,0)$ to $(3,0,0)$, followed by the circular arc from $(3,0,0)$ to $(0,3,0)$, as shown in the figure below. Find the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ with vector field $\mathbf{F}(x,y,z) = ye^x\mathbf{i} + e^x\mathbf{j} + 2z\mathbf{k}$.



Strategy: The curve C is tedious to work with. We can replace it with a simpler one if $\vec{F}$ conservative

Check that $\vec{F}$ conservative: set

$$\vec{F} = \langle ye^x, e^x, 2z \rangle \equiv \langle f, g, h \rangle$$

Then $f_y = e^x = g_x$, $f_z = 0 = h_x$

$g_z = 0 = h_y$

Thus $\vec{F}$ conservative

Integral along a line: we have $\vec{PQ} = \langle 0, 3, -2 \rangle$. Thus define

$$C_1 : \{\vec{r}(t) = \langle 0, 3t, 2-2t \rangle;\ 0 \leq t \leq 1\},$$

i.e. $C_1 \equiv$ segment from P to Q.

Then

$$I \equiv \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r}$$

Computation of the line integral with

$\vec{F} = \langle y e^{x}, e^{x}, 2z \rangle$

$\vec{r}(t) = \langle 0, 3t, 2-2t \rangle$

$\vec{r}\,'(t) = \langle 0, 3, -2 \rangle$

we get

$$I = \int_0^1 \langle 3t, 1, 2(2-2t) \rangle \cdot \langle 0,3,-2 \rangle \, dt$$

$$= \int_0^1 (3 - 4(2-2t)) \, dt$$

$$= -5 + 8 \int_0^1 t \, dt$$

$$= -5 + 4$$

$$\boxed{I = -1}$$

B

13. Suppose $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ and $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 2$ for every (x,y) in the plane.

Find $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r}$, where C is parameterized by $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j}$, $0 \le t \le 2\pi$.

Strategy We are given

$\operatorname{Curl}(\vec{F}) = 2$

Let R be the region enclosed by C. Then according to Green's theorem,

$$I = \oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{Curl}(\vec{F})\, dA$$

$$= 2 \text{ Area}(R)$$

Region R C delimits

$R =$ Disk centered at $(0,0)$, radius $= 3$

Line integral We get

$$I = 2 \text{ Area}(R) = 2\pi \times 3^2$$

$$\boxed{I = 18\pi}$$

E

$$\overset{f}{}\quad \overset{g}{}\quad \overset{h}{}$$

14. Let $\mathbf{F} = \langle x^2yz,\ xy^2z,\ xyz^2\rangle$. Compute grad(div $\mathbf{F}$) $-$ curl(curl($\mathbf{F}$)).

grad (div($\vec{F}$)) we have

$$\operatorname{div}(\vec{F}) = f_x + g_y + h_z = 6xyz$$

$$\nabla(\operatorname{div}(\vec{F})) = 6\langle yz, xz, xy\rangle$$

curl($\vec{F}$)

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ \partial_x & \partial_y \\ x^2yz & xy^2z \end{matrix}$$

$$= \vec{i}\,(xz^2 - xy^2)$$

$$\vec{j}\,(x^2y - yz^2)$$

$$\vec{k}\,(y^2z - x^2z)$$

Thus

$$\operatorname{curl}(\vec{F}) = \langle xz^2 - xy^2,\ x^2y - yz^2,\ y^2z - x^2z\rangle$$

Curl(Curl($\vec{F}$)) we have found

$$\text{Curl}(\vec{F}) = \langle xz^2 - xy^2,\ x^2y - yz^2,\ y^2z - x^2z \rangle$$

Thus

Curl(Curl($\vec{F}$)) =

$$\left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \partial_x & \partial_y & \partial_z & \partial_x & \partial_y \\ xz^2 - xy^2 & x^2y - yz^2 & y^2z - x^2z & xz^2 - xy^2 & x^2y - yz^2 \end{array}\right.$$

$$= \vec{i}\,(2yz + 2yz)$$
$$\vec{j}\,(2zx + 2zx)$$
$$\vec{k}\,(2xy + 2xy)$$

We get

$$\text{Curl}(\text{Curl}(\vec{F})) = 4\langle yz, xz, xy \rangle$$

Conclusion
$$\boxed{\nabla\,\text{div}(\vec{F}) - \text{Curl}(\text{Curl}\,\vec{F}) = 2\langle yz, xz, xy \rangle}$$

A

15. If surface S is parametrized by $\mathbf{r}(u, v) = \langle u, v, uv^2 \rangle$, then the equation of the plane tangent to S at (1, 2, 4) is

Strategy This surface is in fact of the form

$$z = f(x,y) \equiv xy^2$$

We thus use the following formula for the tangent plane:

$$z = f(a,b) + f_x (x-a) + f_y (y-b)$$

Derivatives

$$f_x(x,y) = y^2 \Rightarrow f_x(1,2) = 4$$

$$f_y(x,y) = 2xy \Rightarrow f_y(1,2) = 4$$

Equation We get

$$z = 4 + 4(x-1) + 4(y-2)$$

$$\boxed{4x + 4y - z - 8 = 0}$$

E

16. Let S be the portion of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 3$. Find

$$\iint_S (z+1)\, dS. \equiv I$$

Surface parametrization Since we are based on the cylinder $x^2+y^2=3$, we shall use cylindrical coordinates. We thus set $u = r$, $\theta = v$. The surface S is decribed by

$$\{ < u\cos(v),\ u\sin(v),\ \overbrace{u^2 \sin(v)\cos(v)}^{\frac{1}{2}u^2\sin(2v)} >;$$

$$0 \leq u \leq \sqrt{3},\quad 0 \leq \theta \leq 2\pi \}$$

Surface element We have

$$\vec{E}_u = < \cos(v),\ \sin(v),\ u\sin(2v) >$$

$$\vec{E}_v = < -u\sin(v),\ u\cos(v),\ u^2\cos(2v) >$$

$$\vec{E}_u \times \vec{E}_v = < u^2(\sin(v)\cos(2v) - \sin(2v)\cos(v)),$$

$$-u^2(\sin(2v)\sin(v) + \cos(v)\cos(2v)),\ u >$$

$$\vec{E}_u \times \vec{E}_v = < u^2\sin(v),\ -u^2\cos(v),\ u >$$

$$|\vec{E}_u \times \vec{E}_v| = (u^4 + u^2)^{1/2} = u\,(1+u^2)^{\frac{1}{2}}$$

Surface integral we compute

$$I = \iint_S (z+1)\, dS$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \left(\frac{1}{2}u^2\sin(2v)+1\right) u\,(1+u^2)^{\frac{1}{2}}\, du\, dv$$

$$= \frac{1}{2}\overbrace{\int_0^{2\pi}\sin(2v)\, dv}^{0}\int_0^{\sqrt{3}} u^3(1+u^2)^{\frac{1}{2}}\, du$$

$$+ \frac{2\pi}{2}\int_0^{\sqrt{3}} (2u)(1+u^2)^{\frac{1}{2}}\, du$$

$$= \pi \times \frac{2}{3}(1+u^2)^{3/2}\Big|_0^{\sqrt{3}}$$

$$= \frac{2}{3}\pi\,(4^{3/2}-1)$$

$$= \frac{2}{3}\pi\,(8-1)$$

$$\boxed{I = \frac{14\pi}{3}}$$

B

⑯- second solution

Integral with explicit z we have $z = xy$. Thus we have

$$z_x = y \qquad z_y = x$$

and

$$I = \iint_{x^2+y^2\leq 3} (xy+1)\,(1+z_x^2+z_y^2)^{\frac{1}{2}}\,dx\,dy$$

$$= \iint_{x^2+y^2\leq 3} (xy+1)\,(1+x^2+y^2)^{\frac{1}{2}}\,dx\,dy$$

Polar coordinates The domain becomes

$$D = \{\, 0\leq\theta\leq 2\pi,\ \ 0\leq r\leq\sqrt{3}\,\}$$

and same expression as in previous page

$$I = \int_0^{2\pi}\int_0^{\sqrt{3}} (r^2\sin\theta\cos\theta+1)\,(1+r^2)^{\frac{1}{2}}\,r\,d\theta\,dr$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \left(\frac{1}{2}u^2\sin(2v)+1\right) u\,(1+u^2)^{\frac{1}{2}}\,du\,dv$$

$$= \frac{4\pi}{3}$$

This solution is shorter!

17. Consider the curve $C : \mathbf{r}(t) = \langle \cos(t),\ \sin(t),\ 1 - \cos(t) - \sin(t) \rangle$, $\quad 0 \le t \le 2\pi$, which is the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $x + y + z = 1$.

If $\mathbf{F} = \langle y + \sin(x),\ z + \sin(y),\ x + \cos(z) \rangle$, then $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} =$

Strategy If we compute the line integral directly, we will have intractable integrals of the form $\int \sin(\sin(t))\, dt$. It seems simpler to use Stokes theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \operatorname{Curl}(\vec{F}) \cdot \vec{n}\, dS,$$

where

$$S = \{ z = 1 - x - y \ ; \ x^2 + y^2 \le 1 \}$$

Curl $\vec{F}$ we have

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ y + \sin(x) & z + \sin(y) & x + \cos(z) \end{vmatrix}$$

$$\boxed{\operatorname{Curl}(\vec{F}) = -\langle 1, 1, 1 \rangle}$$

Surface integral we compute

$$\int_S \operatorname{curl}(\vec{F}) \cdot \vec{n}\, dS$$

normal to $x+y+z=1$

$$= \int_S -\langle 1,1,1\rangle \langle 1,1,1\rangle \, dS$$

$$= -\int_{\{x^2+y^2 \le 1\}} 3 \, dx \, dy$$

$$= -3 \times \pi$$

Thus

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = 3\pi}$$

A

$< f, g, h >$

18. Find the flux of $\mathbf{F} = \langle z^2, xy, y^2 \rangle$ out of the box with six faces: $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 0$, and $z = 3$.

Strategy: In order to avoid the tedious parameterization of the boundary, we use the divergence theorem

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_D \mathrm{Div}(\vec{F}) \, dV$$

Domain: We have

$$D = \{ 0 \leq x \leq 1, \ 0 \leq y \leq 2, \ 0 \leq z \leq 3 \}$$

Divergence: We compute

$$\mathrm{Div}(\vec{F}) = f_x + g_y + h_z = 0 + x + 0 = x$$

Flux: We compute

$$\iiint_D \mathrm{Div}(\vec{F}) \, dV = \int_0^1 \int_0^2 \int_0^3 x \, dz \, dy \, dx$$

$$= 2 \times 3 \times \int_0^1 x \, dx = 3$$

$$\boxed{\text{Flux} = 3}$$

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19. Let S be the upper hemisphere of $x^2+y^2+z^2=4$ with normal vector pointing toward the origin, and $\mathbf{F}=z\dfrac{\mathbf{x}}{|\mathbf{x}|}$ where $\mathbf{x}$ denotes the vector $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=\langle x,y,z\rangle$.

Compute $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S}$. $\equiv I$

Strategy: Since $\vec{F} = z \times$ radial function, $\operatorname{div}(\vec{F})$ should not be hard to compute. We thus use the divergence theorem. Moreover the normal vectors are assumed to point inward. We thus get

$$I = \iint_S \vec{F}\cdot d\vec{S} = -\iiint_V \operatorname{div}\vec{F}\, dV,$$

where $V \equiv$ Volume enclosed by S.

Divergence: Let $\vec{G} = \dfrac{\vec{x}}{|\vec{x}|} = \langle g_1, g_2, g_3\rangle$

we get

$$\operatorname{div}(\vec{F}) = \partial_x(z\,g_1) + \partial_y(z\,g_2) + \partial_z(z\,g_3)$$

$$= z\operatorname{div}(\vec{G}) + g_3$$

$$= z \times \frac{2}{|\vec{x}|} + \frac{z}{|\vec{x}|}$$

Recall:
$$\operatorname{div}\left(\frac{\vec{x}}{|\vec{x}|^p}\right) = \frac{3-p}{|\vec{x}|^p}$$

We have thus obtained

$$\boxed{\operatorname{div}(\vec{F}) = \frac{3z}{|\vec{x}|}}$$

Integral It seems natural to use spherical coordinates. We get

$$I = -\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} \frac{3\rho\cos(\varphi)}{\rho}\,\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

$$= -6\pi \int_0^{\pi/2} \sin(\varphi)\cos(\varphi)\,d\varphi \int_0^2 \rho^2\,d\rho$$

$$= -3\pi \int_0^{\pi/2} \sin(2\varphi)\,d\varphi \times \left.\frac{\rho^3}{3}\right|_0^2$$

$$= -8\pi \times \frac{1}{2}\left(-\cos(2\varphi)\Big|_0^{\pi/2}\right)$$

$$\boxed{I = -8\pi}$$

A

20. Consider the vector field $\mathbf{F} = \dfrac{\mathbf{x}}{|\mathbf{x}|^3}$ where $\mathbf{x}$ denotes the vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \langle x, y, z\rangle$. Which of the following are true?

(i) $\operatorname{div}(\mathbf{F}) = 0$ on its maximal domain of definition.

(ii) $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$ on its maximal domain of definition.

(iii) $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$ for any closed surface on which $\mathbf{F}$ is defined.

(iv) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ on any simple, closed, smooth curve on which $\mathbf{F}$ is defined.

(i) $\operatorname{Div}\left(\dfrac{\vec{x}}{|\vec{x}|^p}\right) = \dfrac{3-p}{|\vec{x}|^p} = 0$ if $p = 3$

True

(ii) It has been shown in the book (Section 17.5 Example 5) that

$\vec{F} = \nabla \varphi$, with $\varphi = -\dfrac{1}{|\vec{x}|}$

Hence $\operatorname{Curl}(\vec{F}) = \nabla \times \nabla\varphi = 0$

True

(iii) One would like to use the divergence theorem, which states

$$\iint_S \vec{F} \cdot \vec{dS} = \iiint_V \overbrace{\operatorname{div}(\vec{F})}^{0} \, dV = 0 . \quad (1)$$

However, we are missing the assumptions on S (oriented) and V (connected + simply connected). Hence we cannot claim that (1) is true.

False

(iv) Since C is simple, smooth and closed, and $\vec{F}$ is conservative, we have

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

True

D