# Midterm 1 - Spring 22 - Solutions

**1.** Find the plane tangent to the surface $z^2 = 2xy$ at $\overbrace{(1,2,-2)}^{(a,b,c)}$.

Write the equation as

$$\overbrace{z^2 - 2xy}^{\equiv F(x,y,z)} = 0$$

Then apply Def 9 (slides, functions of several variables) for the tangent plane:

$$\nabla F(a,b,c) \cdot \langle x-a,\ y-b,\ z-c \rangle = 0$$

Here

$$\nabla F(x,y,z) = \langle -2y,\ -2x,\ 2z \rangle$$

At $(1,2,-2)$ we get

$$\nabla F(1,2,-2) = -\langle 4,\ 2,\ 4 \rangle$$

We don't need more computations: the normal to the tangent plane is $\langle 2, 1, 2 \rangle$, and the only possibility in A-E is

$$\boxed{2x + y + 2z = 0}$$

C

**2.** Find the absolute maximum value, $M$, and the absolute minimum value, $m$, of the function $f(x,y) = x^2 + y^2 - 4y + 4$ on the closed disk $\underbrace{\{(x,y) : x^2 + y^2 \leq 16\}}_{R}$.

This is a computation for max-min into a region $R$, as given in Prop 14 (slides on functions of several variables).
We will follow that recipe.

① Inside $R$ compute

$$\nabla f = \langle 2x, 2y-4 \rangle$$

The critical point is $(0,2)$, which belongs to $R$. Moreover

$$f(0,2) = 0^2 + 2^2 - 4 \times 2 + 4$$

$$\Rightarrow f(0,2) = 0$$

② Boundary of $R$ The boundary is the circle

$$C = \{(x,y);\ x^2 + y^2 = 16\}$$

If $(x,y)$ belongs to $C$, then

$$f(x,y) = \overbrace{x^2+y^2}^{=16 \text{ or } C} - 4y + 4$$

$$= 20 - 4y \equiv g(y)$$

We have to study this function for $-4 \leq y \leq 4$. We get two points of interest:

$(0,-4)$, with $f(0,-4) = g(-4) = 36$

$(0,4)$, with $f(0,4) = g(4) = 4$

3 Gathering the 3 points of interest, we get

$$\boxed{m = 0, \qquad M = 36}$$

A

**3.** Consider the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$. Which of the following statements is true?

$f(x,y)$

A. The limit does not exist, because the path-restricted limit approaching $(0,0)$ along the diagonal $y = x$ does not exist.

B. The limit does not exist, even though the path-restricted limits approaching $(0,0)$ along the $x$-axis and the $y$-axis are both 0.

C. The limit does not exist, because the path-restricted limits approaching $(0,0)$ along the $x$-axis and the $y$-axis are different.

D. The limit is 0, and the limit along any path approaching $(0,0)$ is also 0.

E. The limit is 0, because the path-restricted limit approaching $(0,0)$ along the diagonal $y = x$ is 0.

F. The limit is 0, even though the path-restricted limits approaching $(0,0)$ along the $x$-axis and the $y$-axis are different.

Some of those statements can be quickly eliminated

A On the diagonal $y=x$,

$$f(x,x) = \frac{x^2}{x^2+x^2} = \frac{1}{2} \xrightarrow{x\to 0} \frac{1}{2}$$

The limit exists

C If $x=0$, $y\neq 0$, $f(0,y)=0$

If $x\neq 0$, $y=0$, $f(x,0)=0$

Both limits are 0

E Wrong according to A

F Wrong according to C, + absurd statement

We are thus left with (B) or (D).

Let us look at limits along 2 different lines

(i) $y = x$ We have seen

$$\lim_{y=x,\ x,y\to 0} f(x,y) = \frac{1}{2}$$

(ii) $y = 2x$ We have

$$f(x,2x) = \frac{2x^2}{x^2+4x^2} = \frac{2}{5} \xrightarrow{x\to 0} \frac{2}{5}$$

Thus

$$\lim_{y=2x,\ x,y\to 0} f(x,y) = \frac{2}{5}$$

Items (i) and (ii) give two different limits $\Rightarrow$ no limit at $(0,0)$

(B)

**4.** Identify the surface that does **not** contain the curve

$$\vec{r}(t) = \langle \cos t, -\cos t, \sin t\rangle \equiv \langle r_1(t), r_2(t), r_3(t)\rangle$$

A. Plane: $x + y = 0$

B. Circular cylinder: $y^2 + z^2 = 1$

C. Ellipsoid: $\frac{x^2}{2} + \frac{y^2}{2} + z^2 = 1$

D. Circular cylinder: $x^2 + y^2 = 1$

E. Ellipsoid: $\frac{x^2}{3} + \frac{2y^2}{3} + z^2 = 1$

F. Circular cylinder: $x^2 + z^2 = 1$

we shall just check that $\vec{r}(t)$ satisfies (or not) the surface eq.

A $\quad r_1(t) + r_2(t) = \cos t - \cos t = 0$

$\Rightarrow \vec{r}(t) \in$ plane

B $\quad (r_2(t))^2 + (r_3(t))^2 = \cos^2 t + \sin^2 t = 1$

$\Rightarrow \vec{r}(t) \in$ cylinder

C $\quad \frac{(r_1(t))^2}{2} + \frac{(r_2(t))^2}{2} + (r_3(t))^2$

$= \frac{\cos^2 t}{2} + \frac{\cos^2 t}{2} + \sin^2 t = 1$

$\Rightarrow \vec{r}(t) \in$ ellipsoid

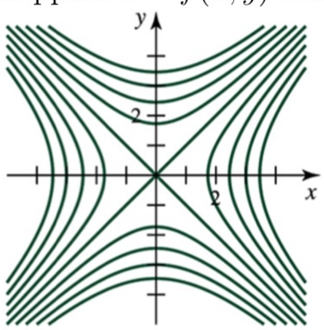
D $(r_1(t))^2 + (r_2(t))^2$

$= 2(\cos(t))^2 \neq 1$ (e.g for $t = \pi/2$ this is 0)

Thus $\boxed{\bar{r}'(t) \notin \text{cylinder}}$

D

**5.** Suppose $z = f(x, y)$ has the following level curves:




The surface formed by the graph of $f$ could be which of the following?

A. Plane

B. Hyperbolic paraboloid

C. Hyperboloid of two sheets

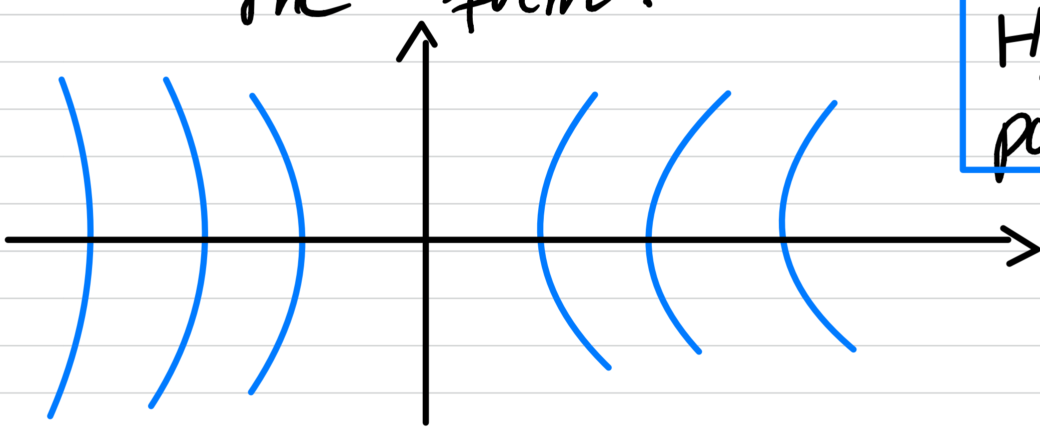
D. Ellipsoid

E. Elliptic paraboloid

F. Elliptic cone

(i) A, D, E, F cannot have hyperbolas as level curves

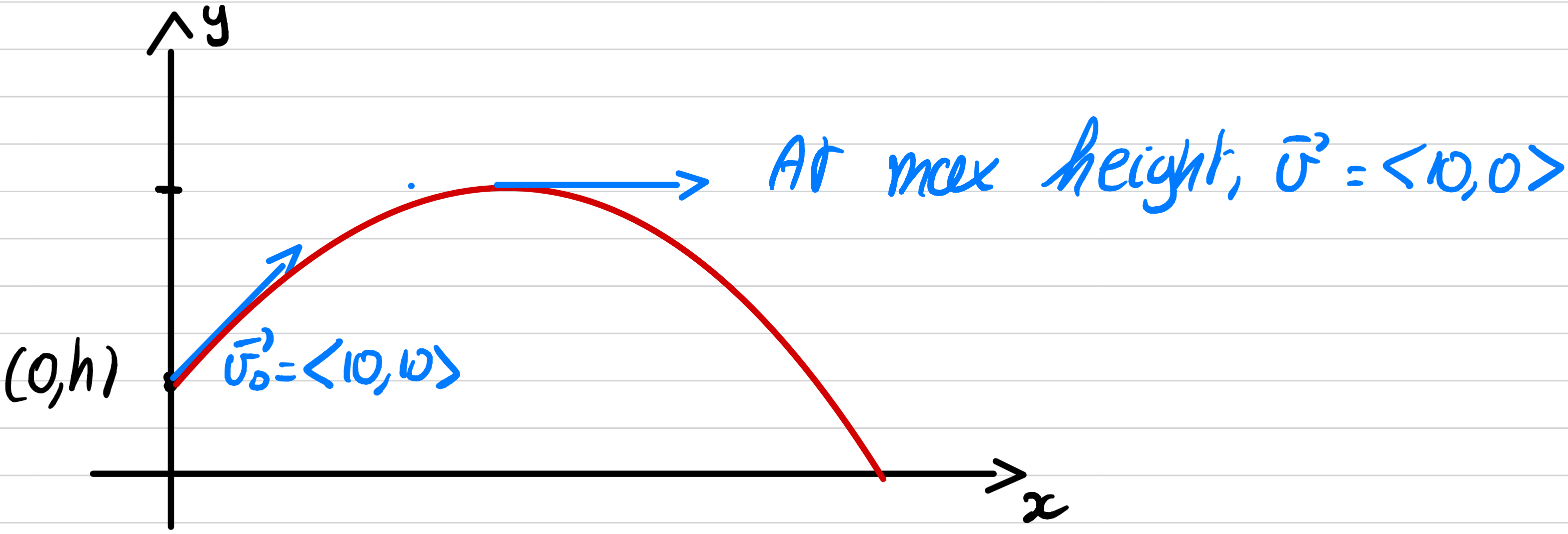
(ii) An hyperboloid of 2 sheets would have level curves of the form:



$\boxed{\text{Hyperbolic paraboloid}}$ is thus the only possibility

B

**6.** A ball is launched from an initial location of $(0, h)$, with initial velocity vector $\langle 10, 10\rangle$. Use the constant $g > 0$ for the acceleration due to gravity, and assume the gravitational force points in the direction of the negative $y$-axis. Determine the location of the ball when it is at its maximum height.




Acceleration

$$\vec{a}(t) = \langle 0, -g\rangle$$

Velocity

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(s)\, ds$$

$$= \langle 10, 10 - gt\rangle$$

Hence $\boxed{v_2(t) = 0 \text{ iff } t = \frac{10}{g}}$

## Position

$$\vec{r}(t) = \vec{r}_0(t) + \int_0^t \vec{v}(s)\,ds$$

$$= \langle 0, h \rangle + \int_0^t \langle 10, 10 - gs \rangle\, ds$$

$$= \left\langle 10t, \; h + 10t - \frac{g}{2}t^2 \right\rangle$$

For $t = \dfrac{10}{g}$ we get

$$\vec{r}(t) = \left\langle \frac{100}{g}, \; h + \frac{100}{g} - \frac{50}{g} \right\rangle$$

$$\boxed{\vec{r}'(t) = \left\langle \frac{100}{g}, \; \frac{50 + hg}{g} \right\rangle}$$

D

7. The line $\ell$ passes through the points $\underbrace{(1,1,1)}^{P_1}$ and $\underbrace{(2,0,1)}^{P_2}$. The plane $Q$ contains the points $\underbrace{(0,0,0)}_{R_1}$, $\underbrace{(1,2,2)}_{R_2}$, and $\underbrace{(1,0,1)}_{R_3}$. Find the intersection point of $\ell$ and $Q$.

Parametric equation for $\ell$ According to Prop 8 (Slides on vectors & geometry) we have

$$\vec{v} = \overrightarrow{P_1P_2} = \langle 1, -1, 0 \rangle$$

and $\ell$ is given by

$$\ell = \{ \langle 1,1,1 \rangle + t \langle 1, -1, 0 \rangle ; \ t \in \mathbb{R} \}$$

$$\ell = \{ \langle 1+t, 1-t, 1 \rangle ; \ t \in \mathbb{R} \}$$

Normal for $Q$ we have

$$\overrightarrow{R_1R_2} = \langle 1, 2, 2 \rangle \qquad \overrightarrow{R_1R_3} = \langle 1, 0, 1 \rangle$$

Thus normal direction is

$$\vec{n} = \overrightarrow{R_1R_2} \times \overrightarrow{R_1R_3} = \langle 2, 1, -2 \rangle$$

Equation for Q The normal is

$\vec{n} = \langle 2, 1, -2 \rangle$

and $(0,0,0)$ belongs to Q. Thus

$$Q: \quad 2x + y - 2z = 0$$

Intersection For $\underbrace{\vec{r}(t)}_{\langle 1+t,\, 1-t,\, 1 \rangle} \in \ell$ we wish to have

$$2r_1(t) + r_2(t) - 2r_3(t) = 0$$

$$\Leftrightarrow 2(1+t) + (1-t) - 2 \times 1 = 0$$

$$\Leftrightarrow 2 + 2t + 1 - t - 2 = 0$$

$$\Leftrightarrow t = -1$$

The corresponding point is

$$\boxed{\vec{r}(-1) = \langle 0, 2, 1 \rangle}$$

A

**8.** Find the arclength function for

$$\vec{r}(t) = \langle \cos t + t \sin t,\ \sin t - t \cos t,\ t^2 \rangle,$$

giving the length of the curve measured from $(1, 0, 0)$ in the direction of positive orientation.

Initial point

$(1,0,0)$ is obtained for $t=0$.

Arc length According to Def 6 (slides on vector valued functions), we have

$$L(t) = \int_0^t |\vec{r}\,'(s)|\, ds$$

Here we have

$$\vec{r}\,'(t) = \langle -\sin t + \sin t + t\cos t,$$
$$\cos t - \cos t + t\sin t,\ 2t \rangle$$

$$\vec{r}\,'(t) = \langle t\cos t,\ t\sin t,\ 2t \rangle$$

Hence

$$|\vec{r}\,'(t)| = \left(t^2\cos^2 t + t^2\sin^2 t + 4t^2\right)^{\frac{1}{2}}$$
$$= \sqrt{5}\,t$$

Arc length computation. We get

$$L(t) = \int_0^t \sqrt{5}\, s \, ds$$

$$\boxed{L(t) = \frac{\sqrt{5}}{2} t^2}$$

C

**9.** Compute the directional derivative of $f(x, y, z) = x^2y + yz^2$ at $(1, 1, 1)$ in the direction $\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}$.

Gradient

$$\nabla f(x,y,z) = \langle 2xy,\ x^2+z^2,\ 2yz \rangle$$

At $(1,1,1)$ we have

$$\nabla f(1,1,1) = \langle 2, 2, 2 \rangle = 2\langle 1,1,1 \rangle$$

Directional derivative Accading to Prop 6 (functions of several variables),

$$D_u f(a,b,c) = \nabla f(a,b,c) \cdot u$$

If $\vec{u} = \frac{1}{3} \langle 2, 1, 2 \rangle$, then

$$D_u f(1,1,1) = 2\langle 1,1,1 \rangle \cdot \frac{1}{3}\langle 2,1,2 \rangle$$

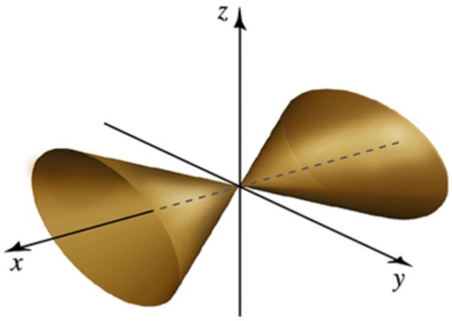
$$= \frac{2}{3} \times 5$$

Hence

$$\boxed{D_u f(1,1,1) = \frac{10}{3}}$$

F

**10.** Which of these equations has a graph like the pictured elliptic cone, with vertex at the origin and opening in the direction of the $x$-axis.




A. $y^2 - 4z^2 - 16x^2 = 1$

B. $y^2 + 4z^2 - 16x^2 = 1$

C. $y^2 + 4z^2 + 16x^2 = 1$

D. $y^2 - 4z^2 + 16x^2 = 0$

E. $y^2 + 4z^2 - 16x^2 = 0$

F. $y^2 - 4z^2 + 16x^2 = 1$

Strategy  Analyze traces

yz-traces  we should get an ellipse, for every $x \in \mathbb{R}$. This ellipse should be reduced to $\{\vec{O}\}$ when $x = 0$. Hence an equation of the form

$$\begin{cases} y^2 + 4z^2 = r(x) \\ r(x) \geq 0 \quad , \quad r(0) = 0 \end{cases}$$

This is in fact enough to determine that the equation is

$$\boxed{y^2 + 4z^2 = 16x^2}$$

E

**11.** Classify all critical points of $f(x,y) = \frac{x^3}{3} - \frac{y^3}{3} + 2xy$, and choose the correct summary from the answer choices below.

Gradient

$$\nabla f(x,y) = \langle x^2 + 2y , -y^2 + 2x \rangle$$

critical points we get a system

$$\begin{cases} x^2 + 2y = 0 \\ -y^2 + 2x = 0 \end{cases}$$

Hence $x = y^2/2$ and substituting we get

$$\frac{y^4}{4} + 2y = 0 \iff y(y^3 + 8) = 0$$

$$\iff y = 0 \text{ or } y = -2$$

Since $x = \frac{y^2}{2}$, we get two critical points

$$\boxed{(0,0) \quad \text{and} \quad (2, -2)}$$

## Second derivatives

Recall that

$$\nabla f(x,y) = \langle \underbrace{x^2+2y}_{f_x}, \underbrace{-y^2+2x}_{f_y} \rangle .$$

Then

$f_{xx}(x,y) = 2x$ $\quad$ $f_{yy}(x,y) = -2y$

$f_{xy}(x,y) = 2$

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = -4xy - 4$

$D(x,y) = -4(xy+1)$

## Summary

| critical pt | $f_{xx}$ | $D(x,y)$ | Classification |
|---|---|---|---|
| $(0,0)$ | $0$ | $-4$ | Saddle |
| $(2,-2)$ | $4$ | $12$ | Min |

D

**12.** Suppose $f$ is a function of $x$, $y$, and $z$, with $f_x(1,1,1) = 1$, $f_y(1,1,1) = 2$, and $f_z(1,1,1) = 3$. If $x = x(t) = t^2$, $y = y(t) = t^3$, and $z = z(t) = t^4$, find $\frac{df}{dt}$ when $t = 1$.

## Chain rule

We apply the chain rule (Thm 3, slides on functions of several variables). This reads

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

$$= f_x \times (2t) + f_y \times (3t^2) + f_z (4t^3)$$

## Application

If $t = 1$ and $(x, y, z) = (1,1,1)$, we obtain

$$\frac{df}{dt} = 1 \times 2 + 2 \times 3 + 3 \times 4$$

$$\boxed{\frac{df}{dt} = 20}$$

C