Midterm 2- Spring 22 - Solutions

1. A cube is given by the region $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, and has density function $\delta = x + yz$. Find the x-coordinate for the cube's center of mass, given that $\iiint_{\text{cube}} \delta \, dV = \frac{3}{4}$.

Mass $\quad m = \frac{3}{4}$

Center

$$\bar{x} = \frac{1}{m} \int_0^1 \int_0^1 \int_0^1 x \overbrace{\delta}^{(x+yz)} \, dx \, dy \, dz$$

$$= \frac{4}{3} \int_0^1 \int_0^1 \int_0^1 (x^2 + xyz) \, dx \, dy \, dz$$

$$= \frac{4}{3} \left(\int_0^1 x^2 \, dx + \int_0^1 x \, dx \int_0^1 y \, dy \int_0^1 z \, dz \right)$$

$$= \frac{4}{3} \left(\frac{1}{3} + \frac{1}{8} \right)$$

$$= \frac{4 \times 11}{3 \times 24}$$

$$\boxed{\bar{x} = \frac{11}{18}}$$

E

2. A helix curve, C, is parametrized by $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, for $0 \le t \le \frac{\pi}{2}$. Compute the line integral

$$\int_C xy \, ds$$

Magnitude of $\bar{r}'$ We have

$\bar{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$

$|\bar{r}'(t)| = (\sin^2(t) + \cos^2(t) + 1^2)^{\frac{1}{2}} = \sqrt{2}$

Line integral

$I = \int_C xy \, ds$

$= \int_0^{\pi/2} \cos(t) \sin(t) \times \sqrt{2} \, dt$

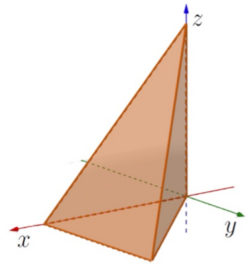
$= \frac{\sqrt{2}}{2} \int_0^{\pi/2} \sin(2t) \, dt$

$= \frac{\sqrt{2}}{4} \quad -\cos(2t) \Big|_0^{\pi/2}$

$\boxed{I = \frac{\sqrt{2}}{2}}$

B

3. Five of these six triple integrals are over the same region of space: the tetrahedron pictured below with vertices at $(0,0,0)$, $(0,0,1)$, $(1,0,0)$ and $(1,1,0)$. One of these triple integrals is over a different region. Which one is different?



A. $\displaystyle\int_0^1\int_0^{1-z}\int_y^{1-z} f(x,y,z)\ dx\ dy\ dz$

B. $\displaystyle\int_0^1\int_0^{1-x}\int_0^{x} f(x,y,z)\ dy\ dz\ dx$

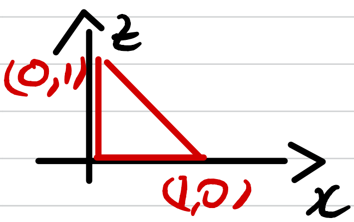
C. $\displaystyle\int_0^1\int_0^{1-z}\int_0^{x} f(x,y,z)\ dy\ dx\ dz$

D. $\displaystyle\int_0^1\int_0^{y}\int_0^{1-x} f(x,y,z)\ dz\ dx\ dy$

E. $\displaystyle\int_0^1\int_0^{1-y}\int_y^{1-z} f(x,y,z)\ dx\ dz\ dy$

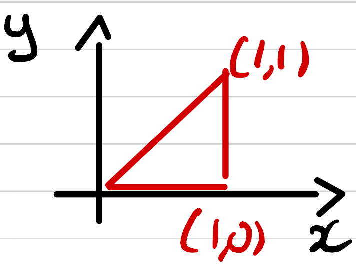
F. $\displaystyle\int_0^1\int_0^{x}\int_0^{1-x} f(x,y,z)\ dz\ dy\ dx$

In the xz-plane the figure is



$$\Rightarrow \begin{cases} 0 \le x \le 1 \\ 0 \le z \le 1-x \end{cases}$$

In the xy-plane we have



$$\Rightarrow \begin{cases} 0 \le x \le 1 \\ 0 \le y \le x \end{cases} \qquad (1)$$

The region in D is such that

$0 \le x \le y$.

This does not respect condition (1).

D

4. Which vector field corresponds to the one pictured here?

A. $\vec{F}(x,y) = \langle 1, -y\rangle$

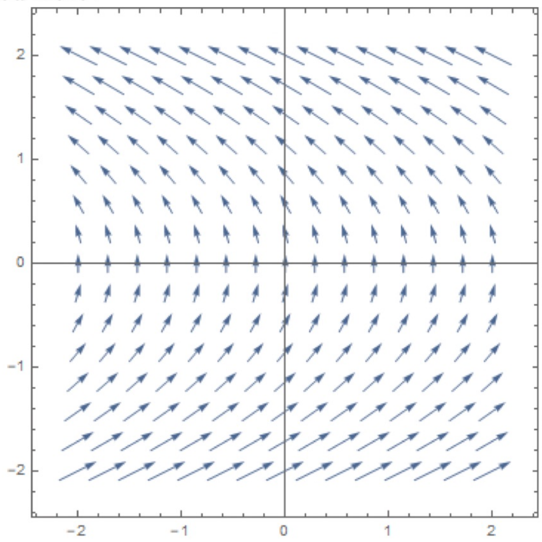
B. $\vec{F}(x,y) = \langle -x, y\rangle$

C. $\vec{F}(x,y) = \langle -y, x\rangle$

D. $\vec{F}(x,y) = \langle 1, y\rangle$

E. $\vec{F}(x,y) = \langle y, 1\rangle$

F. $\vec{F}(x,y) = \langle -y, 1\rangle$



Set $\vec{F} = \langle f, g \rangle$. Then

(i) If $y=0$, we have $f=0$ ($\vec{F}$ is vertical). This restricts our choice to C - E - F.

(ii) Still for $y=0$, we have $g>0$ and g constant. Hence it can only be E or F.

(iii) For $y>0$ we have $f<0$. This is true for F only

F

5. Find $\int_C \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{T}}\, ds$, where $\vec{\boldsymbol{F}}(x, y, z) = \langle ye^z, e^y + xe^z, xye^z \rangle$ on some smooth oriented curve C that goes from $(0, 0, 0)$ to $(-1, 1, 1)$.

A. e

B. -1

C. 0

D. $-e$

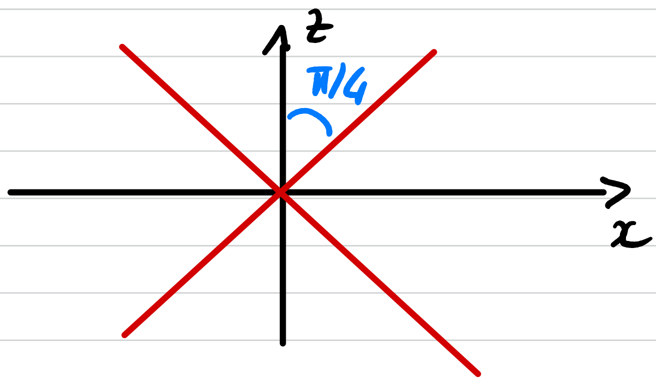
E. 1

F. Impossible to answer without knowing C.

This problem is about conservative vector fields, not part of the program.

6. Choose the triple integral in spherical coordinates that represents the volume of the solid bounded by the cone $z^2 = x^2 + y^2$ and lying between the planes $z = 1$ and $z = 2$. You do not need to compute the volume.

The xz-trace of the cone is $z = \pm x$, which corresponds to $\varphi = \frac{\pi}{4}$



The plane $z = 1$ is $\rho\cos\varphi = 1$

$\iff \rho = \sec\varphi$

The plane $z = 2$ is $\rho = 2\sec\varphi$

Hence the volume is

$$\boxed{\int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\varphi}^{2\sec\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}$$

D

7. Find the absolute maximum value, M, and the absolute minimum value, m, of the function $f(x,y) = x + y$ subject to the constraint $x^2 - xy + y^2 = 1$.

A. $M = 2$ and $m = -2$

B. $M = 1$ and $m = -1$

C. $M = 1$ and $m = -4$

D. $M = 2$ and $m = -1$

E. $M = 4$ and $m = -2$

F. $M = 4$ and $m = -4$

$g(x,y) = x^2 - xy + y^2 - 1$

We use Lagrange's method

1 we have to find x, y, λ s.t.

$$\nabla f = \lambda \nabla g.$$

Here

$\nabla f = \langle 1, 1 \rangle \qquad \nabla g = \langle 2x - y, -x + 2y \rangle$

Hence $\nabla f = \lambda \nabla g$ yields

$$\lambda = 2x - y = -x + 2y$$

$$\Rightarrow \quad 3x = 3y \quad \Rightarrow \quad x = y$$

Replacing in the constraint $g = 0$ we get

$$x^2 - x^2 + x^2 = 1 \iff \boxed{x = \pm 1}$$

2 We have found two critical points:

$(-1,-1)$ and $(1,1)$

Now

$f(-1,1) = -2$, and $f(1,1) = 2$

Thus

$$\boxed{m = -2, \quad M = 2}$$

A

8. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{-1}^{1}(x^2+y^2)^{3/2}\,dz\,dx\,dy \equiv I$

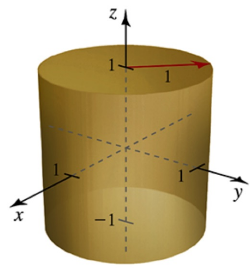
A. $\frac{2\pi}{7}$

B. $\frac{2\pi}{5}$

C. $\frac{4\pi}{5}$

D. $\frac{4\pi}{7}$

E. $\frac{\pi}{}$



Domain in cylindrical coordinates

$$D = \{0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ -1 \le z \le 1\}$$

Integral

$$I = \int_0^{2\pi}\int_0^1\int_{-1}^1 r^3\, r\, dz\, dr\, d\theta$$

$$= 2\pi \times 2 \times \int_0^1 r^4\, dr$$

$$\boxed{I = \frac{4\pi}{5}}$$

C

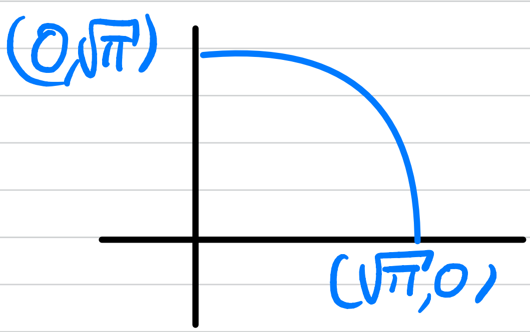
9. $\displaystyle\int_0^{\sqrt{\pi}} \int_0^{\sqrt{\pi - y^2}} \sin(x^2+y^2)\ dx\, dy \equiv I$

Hint: polar

Domain in Cartesian

$D = \{ 0 \le y \le \sqrt{\pi},\quad 0 \le x \le \sqrt{\pi - y^2} \}$

Hence $x^2+y^2 \le \pi$ and $x, y \ge 0$



Domain in polar

$D = \{ 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le \sqrt{\pi} \}$

Integral

$$I = \int_0^{\pi/2} \int_0^{\sqrt{\pi}} \sin(r^2)\, r\, dr\, d\theta$$

of the form $u' \sin(u)$

$$= \frac{\pi}{2} \times \frac{1}{2} \int_0^{\sqrt{\pi}} 2r \sin(r^2)\, dr$$

substitution

$$= \frac{\pi}{4} \left(-\cos(r^2)\right)\Big|_0^{\sqrt{\pi}}$$

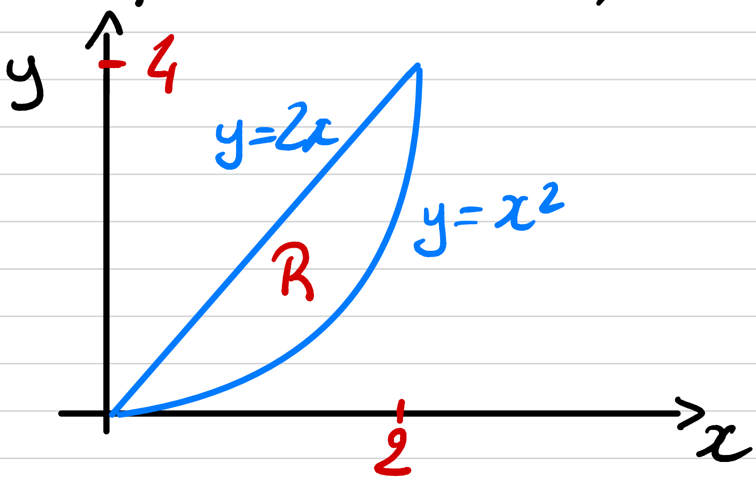
$$\boxed{I = \frac{\pi}{2}}$$

F

10. Change the order of integration for the double integral $\int_0^2 \int_{x^2}^{2x} f(x,y)\, dy\, dx$ $\equiv I$. You do not need to compute the integral.

Domain

$$R = \{ 0 \le x \le 2, \quad x^2 \le y \le 2x \}$$



Alternative expression for R

$$R = \{ 0 \le y \le 4, \quad \frac{y}{2} \le x \le \sqrt{y} \}$$

Thus

$$\boxed{I = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} f(x,y)\, dx\, dy}$$

F

11. Given the force field $\vec{F}(x, y, z) = \langle y, z, x \rangle$, find the work required to move an object along the straight line segment from $\underbrace{(0, 0, 0)}_{P}$ to $\underbrace{(2, 3, 4)}_{Q}$.

Parametrizing the segment

$\vec{PQ} = \langle 2, 3, 4 \rangle$

Hence

$C : \{ \vec{r}(t) = \langle 2t, 3t, 4t \rangle ; 0 \leq t \leq 1 \}$

Work

$6 + 12 + 8$

$$W = \int_0^1 \langle 3t, 4t, 2t \rangle \cdot \overbrace{\langle 2, 3, 4 \rangle}^{\vec{r}'(t)} \, dt$$

$$= \int_0^1 26\, t \, dt$$

$$\boxed{W = 13}$$

12. Use Green's Theorem to evaluate $\int\limits_C x\,dx + (x^2+y^2)\,dy$ where C is the boundary of the rectangle with vertices $(0,0)$, $(2,0)$, $(2,3)$, and $(0,3)$, oriented counterclockwise.

Green's theorem is not included in the program