MIDTERM 1 - SPRING 19

(1) $\vec{n} = \langle 1, -2, 2 \rangle$ is the normal to the plane

$P(-1, 1, 2)$ is a point on the line

Equation for the line:

$\langle x, y, z \rangle = \langle -1, 1, 2 \rangle + t \langle 1, -2, 2 \rangle$

$= \langle -1+t, 1-2t, 2+2t \rangle$

This intersects the yz-plane if $x=0$, i.e $t=1$. Thus the point on the line is

$\langle 0, -1, 4 \rangle$

(E)

② The normal to the plane is perpendicular to both

$\vec{n}_1 = \langle 2, 1, -2 \rangle$, $\vec{n}_2 = \langle 1, 0, 3 \rangle$

Thus a normal is given by

$\vec{n}_1 \times \vec{n}_2 = \langle 3, -8, -1 \rangle \equiv \vec{n}$

The equation of the plane, with $P_0(1, -1, 2)$ is

$\vec{n} \cdot \overrightarrow{P_0P} = 0$

$\Leftrightarrow \langle 3, -8, -1 \rangle \langle x-1, y+1, z-2 \rangle = 0$

$\Leftrightarrow \underline{\underline{3x + 8y - z = 9}}$

B

(3) The cylinder can be parametrized as

x = free variable

$y = \cos(t)$, $z = \sin(t)$, $t \in [0, 2\pi]$

Thus on the plane $x+y+2z=3$ we have

$x = 3 - \cos(t) - 2\sin(t)$

We get a curve of the form

$\vec{r}(t) = \langle 3 - \cos(t) - 2\sin(t), \cos(t), \sin(t) \rangle$

B

(4) Let $\vec{r}(t) = \langle t, \frac{t^2}{2}, \frac{t^3}{3} \rangle$

We use the formula

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)^3|}$$

Next we have

$\vec{r}'(t) = \langle 1, t, t^2 \rangle$ $\vec{r}''(t) = \langle 0, 1, 2t \rangle$

$\vec{r}'(t) \times \vec{r}''(t) = (t^2, -2t, 1)$

Therefore, for $t=1$ we get

$\vec{r}'(t) \times \vec{r}''(t) = (1, -2, 1)$

$|\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{6}$

$\vec{r}'(t) = \langle 1, 1, 1 \rangle$, $|\vec{r}'(t)| = \sqrt{3}$

and

$$\kappa(1) = \frac{\sqrt{6}}{(\sqrt{3})^3} = \frac{\sqrt{2}}{3}$$

C

(5) We have $\vec{r}(t) = \langle 3t, 4\sin(t), 4\cos(t) \rangle$

The length of the curve from 0 to α is

$$s(\alpha) = \int_0^\alpha |\vec{r}'(t)|\, dt$$

$$= \int_0^\alpha |\langle 3, 4\cos(t), -4\cos(t) \rangle|\, dt$$

$$= \int_0^\alpha \sqrt{9+16}\, dt$$

$$= 5\alpha$$

Thus $s(\alpha) = 20 \iff 5\alpha = 20$

$$\iff \alpha = 4$$

D

(6) We have

$\vec{a} = \langle 6t-2, -1/t^2, 0 \rangle$, $\vec{v}(1) = \langle 1,1,1 \rangle$

Thus for $t \geq 1$,

$$\vec{v}(t) = \vec{v}(1) + \int_1^t \vec{a}(s)\,ds$$

$$= \langle 1,1,1 \rangle + \int_1^t \langle 6s-2, -1/s^2, 0 \rangle\,ds$$

$$= \langle 1,1,1 \rangle + \langle 3s^2-2s, \tfrac{1}{s}, 0 \rangle \Big|_1^t$$

$$= \langle 3t^2-2t, \tfrac{1}{t}, 1 \rangle$$

Then

$$\vec{r}(t) = \vec{r}(1) + \int_1^t \vec{v}(s)\,ds$$

$$= \langle 0,0,3 \rangle + \int_1^t \langle 3s^2-2s, \tfrac{1}{s}, 1 \rangle\,ds$$

$$= \langle 0,0,3 \rangle + \langle s^3-s^2, \ln(s), s \rangle \Big|_1^t$$

$$= \langle t^3-t^2, \ln(t), t+2 \rangle$$

and

$$\vec{r}(2) = \langle 4, \ln(2), 4 \rangle$$

$$|\vec{r}(2)| = \underline{\underline{(32 + (\ln(2))^2)^{1/2}}}$$

C

7 $f(x,y) = \sqrt{x^2+1} - 2y$

The level curves are defined by

$$\sqrt{x^2+1} - 2y = z_0$$

This can also be written as

$$2y + z_0 = \sqrt{x^2+1}$$

$$\Rightarrow \quad (2y+z_0)^2 = x^2+1$$

$$\Rightarrow \quad 4y^2 - x^2 = \text{lower order terms}$$

We get a hyperbola

A

⑧ Let $f(x,y,z) = \dfrac{xz}{\sqrt{y^2 - z}}$

Then

$$f_x = \frac{z}{\sqrt{y^2-z}} = z\,(y^2-z)^{-1/2}$$

$$f_{xy} = -\frac{1}{2} \times 2yz \quad (y^2-z)^{-3/2}$$
$$= -yz \quad (y^2-z)^{-3/2}$$

$$f_{xyz} = -y\,(y^2-z)^{-3/2} - \frac{3}{2}\;yz\;(y^2-z)^{-5/2}$$
$$= -\frac{y\,(2y^2+z)}{2(y^2-z)^{5/2}}$$

Thus

$$f_{xyz}(1,2,3) = -\frac{2\;(8+3)}{2 \times 1^{5/2}}$$

$$f_{xyz}(1,2,3) = -11$$

(B)

(9) $z = e^r \cos\theta$ with

$r = 12st \qquad \theta = (s^2 + t^2)^{\frac{1}{2}}$

Then according to the chain rule,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$= e^r \cos\theta \times 12t - e^r \sin\theta \times s\,(s^2+t^2)^{-\frac{1}{2}}$$

$$= e^r \left\{ 12t\cos\theta - s\sin\theta\,(s^2+t^2)^{-\frac{1}{2}} \right\}$$

(A)

(10) $f(x,y) = x^2 y + e^{xy} \sin(y) + 15$

The direction of maximal increase is given by the gradient. We compute

$$f_x = 2xy + y e^{xy} \sin(y)$$
$$f_y = x^2 + e^{xy} (x \sin(y) + \cos(y))$$

Thus

$$\nabla f(1,0) = \langle f_x(1,0), f_y(1,0) \rangle$$
$$= \langle 0, 2 \rangle$$

The corresponding unit vector is

$$\frac{\nabla f(1,0)}{|\nabla f(1,0)|} = \langle 0, 1 \rangle = \underline{\underline{\vec{j}}}$$

B

(11) $z = f(x,y) = x - \frac{y^2}{2}$

We use the equation for a tangent plane for $z = f(x,y)$, at $(x,y) = (a,b)$

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b) \quad (1)$$

Here

$f_x = 1$ $f_y = -y$

Thus for $(a,b) = (2,4)$ and $z = -6$

$f(2,4) = -6$

$f_x(2,4) = 1$ $f_y(2,4) = -4$

and (1) becomes

$$z = 1 \times (x-2) - 4(y-4) - 6$$

$$\Leftrightarrow z = x - 4y + 8$$

$$\Leftrightarrow \underline{\underline{-x + 4y + z = 8}}$$

(D)

(12) Local minimums are not included for our Midterm 1