Another rational function example (1)

If $x = y = 0$, then
$x^2 - y^2 = 0$
$x + y = 0$

Function:

$$f(x, y) = \frac{x^2 - y^2}{x + y}$$

Problem: Continuity at point

$$(0, 0)$$

We will see: $\frac{0}{0}$, but f is continuous at $(0,0)$

Function : $f(x,y) = \dfrac{x^2 - y^2}{x+y}$ $(a,b) = (0,0)$

Limit along line $x = 0$:

$$f(0,y) = \frac{0^2 - y^2}{0 + y} = -y \xrightarrow{y \to 0} 0$$

Limit along line $y = 0$:

$$f(x,0) = \frac{x^2 - 0^2}{x + 0} = x \xrightarrow{x \to 0} 0$$

Rmk : We cannot conclude as in the previous example

$$f(x,y) = \frac{x^2 - y^2}{x+y}$$

Limit along line $x = 4y$

$$f(4y, y) = \frac{16y^2 - y^2}{4y + y} = \frac{15y^2}{5y} = 3y \xrightarrow{y \to 0} 0$$

Limit along $x = 2y^2$

$$f(2y^2, y) = \frac{4y^4 - y^2}{2y^2 + y} = \frac{y^2(4y^2 - 1)}{y(2y+1)}$$

$$= \frac{y(4y^2 - 1)}{2y + 1} \xrightarrow{y \to 0} 0$$

Rmk We have found 4 paths for which the limit is 0
But we still cannot conclude

$$f(x,y) = \frac{x^2-y^2}{x+y}$$

Formula : $a^2 - b^2 = (a-b)(a+b)$

Application: $f(x,y) = \frac{(x-y)(x+y)}{x+y} = x-y \xrightarrow{(x,y)\to(0,0)} 0$

If we set $f(0,0) = 0$, then f becomes **continuous at $(0,0)$**

Another rational function example (2)

Continuity: f is a rational function
$\hookrightarrow$ Continuous wherever it is defined

Definition at point $(0,0)$: We have

$$f(0,0) = \frac{0}{0}$$

This is not well defined, therefore general result cannot be applied

Another rational function example (3)

Two paths: We have

$$\text{Along } x = 0, \quad \lim_{(x,y)\to(0,0),\, x=0} \frac{x^2 - y^2}{x + y} = 0$$

$$\text{Along } y = 0, \quad \lim_{(x,y)\to(0,0),\, y=0} \frac{x^2 - y^2}{x + y} = 0$$

We get the same limit

Partial conclusion:
This is not enough!

Another rational function example (4)

Next steps: Try different paths

- $y = x^2$, $y = x^3$, etc
- Those all give a 0 limit
- This is still not enough

Key remark: If $(x, y) \neq (0, 0)$ we have

$$f(x, y) = \frac{x^2 - y^2}{x + y} = x - y$$

The rhs above is continuous

Conclusion: We have

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0$$

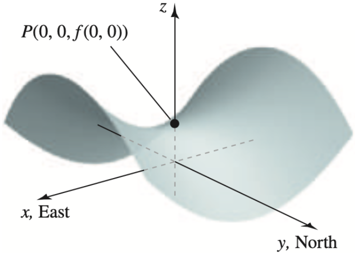
Outline

Motivation

Derivative for functions of 1 variable: Captures the rate of change

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Rate of change in the 2-d case: Can be different in x and y directions
$\hookrightarrow$ Captured by partial derivatives



Partial derivatives

Definition 2.

Consider

- f function of 2 variables

Then we set

$$
\begin{aligned}
f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \\
f_y(x,y) &= \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}
\end{aligned}
$$

Some remarks on partial derivatives

Frozen and live variables:

- In order to compute $f_x(x, y)$
  $\hookrightarrow$ the x variable is alive and the y variable is frozen
- In order to compute $f_y(x, y)$
  $\hookrightarrow$ the y variable is alive and the x variable is frozen

Funny notation: For partial derivatives we also use

$$\frac{\partial f}{\partial x}(x, y) = f_x(x, y), \qquad \frac{\partial f}{\partial y}(x, y) = f_y(x, y)$$

Example of computation (1)

Function:

$$f(x, y) = x^8 y^5 + x^3 y$$

Function $f(x,y) = x^8 y^5 + x^3 y$

Computation for f_x : y frozen, x alive

$$f_x(x,y) = \boxed{8x^7y^5 + 3x^2y}$$

Computation for f_y: y alive, x frozen

$$f_y(x,y) = \boxed{5x^8y^4 + x^3}$$

Example of computation (2)

Recall:

$$f(x,y) = x^8y^5 + x^3y$$

Partial derivative f_x:

$$f_x = 8x^7y^5 + 3x^2y$$

Partial derivative f_y:

$$f_y = 5x^8y^4 + x^3$$

Second example of computation (1)

Function:

$$f(x, y) = e^x \sin(y)$$

Function $f(x,y) = e^x \sin(y)$

f_x : $f_x(x,y) = e^x \sin(y)$

f_y : $f_y(x,y) = e^x \cos(y)$

Second example of computation (2)

Recall:

$$f(x, y) = e^x \sin(y)$$

Partial derivative f_x:

$$f_x = e^x \sin(y)$$

Partial derivative f_y:

$$f_y = e^x \cos(y)$$

Second derivatives

Second derivative f_{xx}, f_{yy}:

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2}, \qquad f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2}$$

Second derivative f_{xy}:

$$f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial x \partial y}$$

Second derivative f_{yx}:

$$f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial y \partial x}$$

Example of second derivatives

Function:

$$f(x,y) = e^x \sin(y)$$

Second derivative f_{xx}:

$$f_{xx} = (f_x)_x = e^x \sin(y)$$

Second derivative f_{xy}:

$$f_{xy} = (f_x)_y = e^x \cos(y)$$

Function $f(x,y) = e^x \sin(y)$

We have seen $f_x(x,y) = e^x \sin(y)$

$f_y(x,y) = e^x \cos(y)$

Computation for f_{xx}

$$f_{xx}(x,y) = (f_x)_x(x,y)$$
$$= (e^x \sin(y))_x = e^x \sin(y)$$

Computation for f_{xy}

$$f_{xy}(x,y) = (f_x)_y(x,y)$$
$$= (e^x \sin(y))_y = e^x \cos(y)$$

Computation f_{yx}

$$f_{yx}(x,y) = (f_y)_x(x,y) = (e^x \cos(y))_x = e^x \cos(y)$$

$f_{xy} = f_{yx}$

Order of derivatives

On our running example: We have

$$f_{yx} = (f_y)_x = e^x \cos(y) = f_{xy}$$

General result (Clairaut's theorem):
For a smooth f, the order of the derivatives does not matter

$$f_{yx} = f_{xy}$$

Example of order of derivatives (1)

Function:

$$f(x, y) = e^{x^2 y}$$

Problem: Check that

$$f_{yx} = f_{xy}$$

Example of order of derivatives (2)

Recall:

$$f(x,y) = e^{x^2 y}$$

Partial derivative f_x:

$$f_x = 2xy\, e^{x^2 y}$$

Partial derivative f_y:

$$f_y = x^2\, e^{x^2 y}$$

Mixed derivatives:

$$f_{yx} = f_{xy} = 2x\left(x^2 y + 1\right) e^{x^2 y}$$

Functions of 3 variables (1)

Basic rule: Functions of 3 variables are handled
$\hookrightarrow$ in the same way as functions of 2 variables

Example:

$$f(x, y, z) = xyz$$

First derivatives:

$$f_x = yz, \qquad f_y = xz, \qquad f_z = xy$$

Function $f(x,y,z) = xyz$

Compute f_x: $(xyz)_x = yz$

In the same way, $f_y = xz$ $f_z = xy$

Second order derivatives

$(xyz)_{xy} = ((xyz)_x)_y = (yz)_y = z$

$(xyz)_{yx} = ((xyz)_y)$

$= (xz)_x = z$ //

We also have $\boxed{f_{xy} = f_{yx}}$ in this case in $\mathbb{R}^3$

Functions of 3 variables (2)

Second derivatives: We have for instance

$$f_{xy} = f_{yx} = z$$

Third derivatives: The only non zero derivatives are

$$f_{xyz} = f_{xzy} = \cdots = f_{zyx} = 1$$