Brief summary of partial derivatives

Def of f_x, f_y, $f_{xy} = (f_x)_y$, f_{xyz} and relation

$$\boxed{f_{xy} = f_{yx}}$$

Example of computation : $f(x,y) = e^{x^2y}$

Then

$$f_x(x,y) = 2xy \, e^{x^2y}$$

$$f_{xy}(x,y) = (f_x)_y(x,y) = 2x\left(y\,e^{x^2y}\right)_y$$

$$= 2x\,(1 + x^2y)\,e^{x^2y}$$

we also have

$$f_y(x,y) = x^2\,e^{x^2y}$$

$$f_{yx}(x,y) = (f_y)_x(x,y) = \left(x^2 e^{x^2y}\right)_x$$

$$= (2x + x^2 \times 2xy)\,e^{x^2y}$$

$$= 2x\,(1 + x^2y)\,e^{x^2y}$$

we have verified $f_{xy} = f_{yx}$

Outline

Chain rule for functions of 1 variable

Usual 1-d chain rule:

$$\underbrace{(f \circ g)'}_{y=y(t)} = \underbrace{f'(g)}_{\frac{dy}{dx}} \underbrace{g'}_{\frac{dx}{dt}}$$

x

Situation: We have

- $y = f(x)$
- $x = g(t)$

Chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}x}{\mathrm{d}t}$$

Chain rule with 1 independent variable

$$\frac{z(t+h) - z(t)}{h}$$

Theorem 3.

Let

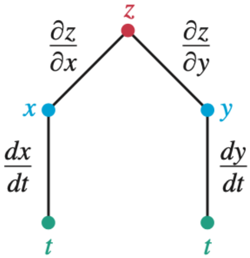
- $z = z(x, y)$
- $x = x(t)$ and $y = y(t)$
- z, x, y differentiable

In the end $z = z(x(t), y(t))$ only depends on t. It is a function of 1 variable

Then

$$\underset{\displaystyle \frac{\mathrm{d}z}{\mathrm{d}t}}{\overset{z'(t)}{\parallel}} = \underset{\displaystyle \frac{\partial z}{\partial x}}{\overset{z_x}{\parallel}} \frac{\mathrm{d}x}{\mathrm{d}t} + \underset{\displaystyle \frac{\partial z}{\partial y}}{\overset{z_y}{\parallel}} \frac{\mathrm{d}y}{\mathrm{d}t}$$

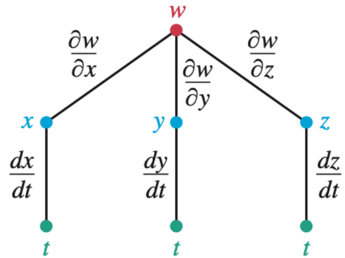
Tree representation of chain rule (2d)



$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

Tree representation of chain rule (3d)

$w(x, y, t)$ and $x = x(t)$, $y = y(t)$, $z = z(t)$



$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}}$$

Example of computation (1)

Functions: We consider

$$z = x^2 - 3y^2 + 20, \quad x = 2\cos(t), \quad y = 2\sin(t)$$

Derivative: We find

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= -16\sin(2t) \end{aligned}$$

Particular value: It $t = \frac{\pi}{4}$, then

$$\frac{\mathrm{d}z}{\mathrm{d}t}\left(\frac{\pi}{4}\right) = -16$$

$z_x = 2x \qquad z_y = -6y$

Function: $z = x^2 - 3y^2 + 20$

$x = 2\cos(t) \qquad y = 2\sin(t)$

We wish to compute $z'(t) = \frac{dz}{dt}$

Applying Thm 3,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= 2x \times (-2\sin(t)) - 6y\,(2\cos(t))$$

$$= 2 \times 2\cos(t) \times (-2\sin(t)) - 6 \times 2\sin(t) \times 2\cos(t)$$

$$= -32\sin(t)\cos(t) \qquad 2\sin(a)\cos(a) = \sin(2a)$$

$$\boxed{\frac{dz}{dt} = -16\sin(2t)}$$

Function: $z = x^2 - 3y^2 + 20$

$x = 2\cos(t) \qquad y = 2\sin(t)$

Another possibility to compute $\frac{dz}{dt}$.

write

$$z(t) = z(x(t), y(t)) = x^2(t) - 3y^2(t) + 20$$

$$= (2\cos(t))^2 - 3(2\sin(t))^2 + 20$$

$$= 4\cos^2(t) - 12\sin^2(t) + 20$$

Then differentiate this expression wrt t

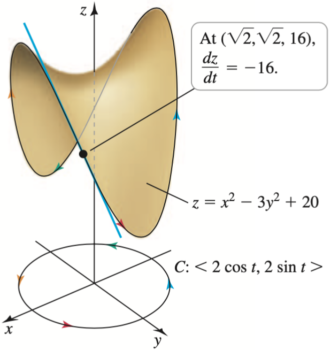
Problem This works on our simple example.
For anything more complicated, we still might want to apply Thm 3.

Example of computation (2)

Other possible strategy:

1. Express $z(x(t), y(t))$ as a function $F(t)$
2. Differentiate as usual

Problem: this becomes impractical very soon.



Implicit differentiation

<u>Example</u> : $F(x,y) = x^2+y^2-1$
The eq $F(x,y)=0 \Leftrightarrow x^2+y^2=1$ defines the unit circle

Theorem 4.

Let $F(x, y)$ be such that

- F differentiable
- The equation $F(x, y) = 0$ defines $y = y(x)$
- $x \mapsto y(x)$ differentiable
- $F_y \neq 0$

Then we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$$

Example of implicit differentiation (1)

Equation:

$$e^y \sin(x) = x + xy$$

Problem: Find

$$\frac{\mathrm{d}y}{\mathrm{d}x}$$

Equation $\underbrace{e^y \sin(x) - x - xy}_{F(x,y)} = 0$

According to Thm 4, if $y = y(x)$ then

$$y'(x) = \frac{dy}{dx} = \frac{-F_x}{F_y}$$

$$\boxed{\frac{dy}{dx} = -\frac{(e^y \cos(x) - 1 - y)}{e^y \sin(x) - x}}$$

Example of implicit differentiation (2)

Reformulation of the equation: $F(x, y) = 0$ with

$$F(x, y) = e^y \sin(x) - x - xy$$

Implicit differentiation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y} = -\frac{e^y \cos(x) - 1 - y}{e^y \sin(x) - x}$$