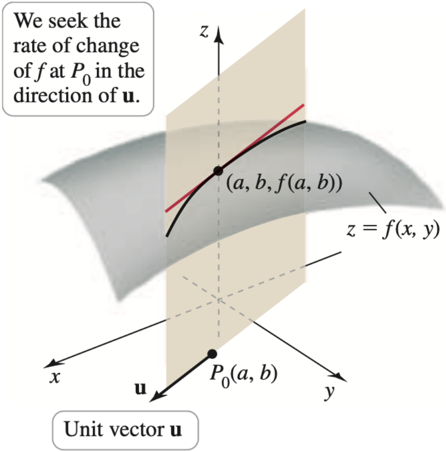
Outline

Objective

Aim: Understand variations of a function
↪ In directions which are not parallel to the axes



Directional derivative

Definition 5.

Let

- f differentiable function at (a, b)
- $\mathbf{u} = \langle u_1, u_2 \rangle$ unit vector in xy-plane

$u_1^2 + u_2^2 = 1$ a $|u| = 1$

Then the directional derivative of f
in the direction of $\mathbf{u}$ at (a, b) is

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

Computation of the directional derivative

Proposition 6.

Let

- f differentiable function at (a, b)
- $\mathbf{u} = \langle u_1, u_2 \rangle$ unit vector in xy-plane

Then the directional derivative of f in the direction of $\mathbf{u}$ at (a, b) is given by

$$D_{\mathbf{u}} f(a, b) = f_x(a, b) u_1 + f_y(a, b) u_2$$

Rmk In particular, if we know f_x and f_y, we can compute all the directional derivatives

Remark: One can also write

$$D_{\mathbf{u}} f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle$$

Example of directional derivative (1)

Function: Paraboloid of the form

$$z = f(x,y) = \frac{1}{4}\left(x^2 + 2y^2\right) + 2$$

Unit vector:

$$\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$|u|^2 = \frac{1}{2} + \frac{1}{2} = 1$
$\Rightarrow$ unit vector

Problem: Compute the directional derivative

$$D_{\mathbf{u}} f(3,2)$$

Function $f(x,y) = \frac{1}{4}x^2 + \frac{1}{2}y^2 + 2 = z$

Vector $u = \langle \underbrace{1/\sqrt{2}}_{u_1}, \underbrace{1/\sqrt{2}}_{u_2} \rangle$

Directional derivative

$$D_u f(x,y) = f_x(x,y)\cdot u_1 + f_y(x,y)\, u_2$$

$$= \frac{1}{2}x \times \frac{1}{\sqrt{2}} + y \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{2}x + y\right)$$

Thus

$$\boxed{D_u f(3,2)} = \frac{1}{\sqrt{2}}\left(\frac{3}{2} + 2\right) = \boxed{\frac{7}{2\sqrt{2}}} \simeq 2.47$$

Note $f(3,2) = \frac{25}{4}$

Example of directional derivative (2)

Function: Paraboloid of the form

$$z = f(x, y) = \frac{1}{4}\left(x^2 + 2y^2\right) + 2$$

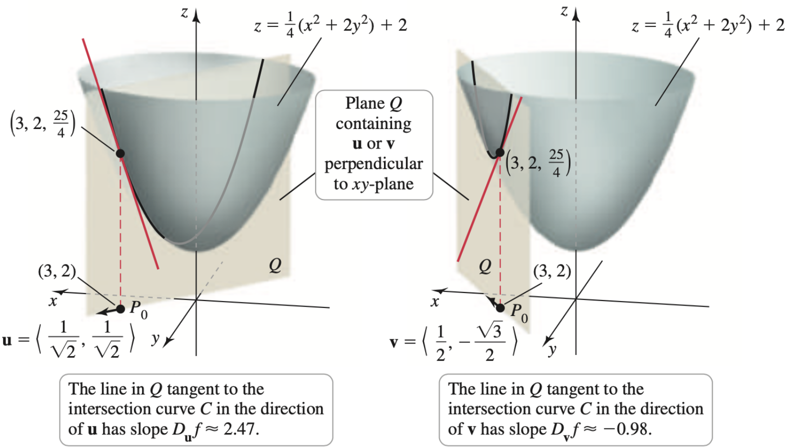
Unit vector:

$$\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Directional derivative: We get

$$D_{\mathbf{u}} f(3, 2) = \left\langle \frac{3}{2}, 2 \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{7}{2\sqrt{2}} \simeq 2.47$$

Example of directional derivative (3)



Gradient

Definition 7.

Let

- f differentiable function at (x, y)

Then the gradient of f at (x, y) is

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

nabla

Example of gradient (1)

Function:

$$f(x,y) = 3 - \frac{x^2}{10} + \frac{xy^2}{10}$$

Problem:

1. Compute $\nabla f(3,-1)$
2. Compute the directional derivative of f
   $\hookrightarrow$ at $(3,-1)$ in the direction of the vector $\langle 3,4 \rangle$

Function $f(x,y) = 3 - \frac{x^2}{10} + \frac{xy^2}{10}$

Gradient

$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

$= \langle -\frac{x}{5} + \frac{y^2}{10}, \frac{xy}{5} \rangle$

At point (3, −1) we get

$\nabla f(3,-1) = \langle -\frac{3}{5} + \frac{1}{10}, -\frac{3}{5} \rangle = \boxed{\langle -\frac{1}{2}, -\frac{3}{5} \rangle}$

In the direction of $u = \langle 3,4 \rangle$ $|u| = \sqrt{3^2+4^2} = 5$

Thus set $\vec{v} = \langle 3/5, 4/5 \rangle$. The deriv is

$D_{\vec{v}} f(3,-1) = \nabla f(3,-1) \cdot \vec{v} = -\frac{1}{2} \times \frac{3}{5} - \frac{3}{5} \times \frac{4}{5} = \boxed{-\frac{39}{50}}$

Example of gradient (2)

Gradient:

$$\nabla f(x, y) = \left\langle -\frac{x}{5} + \frac{y^2}{10}, \frac{xy}{5} \right\rangle$$

Thus

$$\nabla f(3, -1) = \left\langle -\frac{1}{2}, -\frac{3}{5} \right\rangle$$

Example of gradient (3)

Directional derivative: Unit vector in direction of $\langle 3, 4\rangle$ is

$$\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

Thus directional derivative in direction of $\langle 3, 4\rangle$ is

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

We get

$$D_{\mathbf{u}} f(3, -1) = -\frac{39}{50}$$

Interpretation of gradient

$|u|=1$

$D_u f(x,y) = \nabla f(x,y)\cdot u = |\nabla f(x,y)| \cos(\theta)$

Remark: If

- $\mathbf{u}$ is a unit vector
- $\theta \equiv$ angle between $\mathbf{u}$ and $\nabla f(x,y)$

Q: when is $\cos(\theta)$ max?
A: when $\theta = 0$, ie
u parallel to ∇f

Then

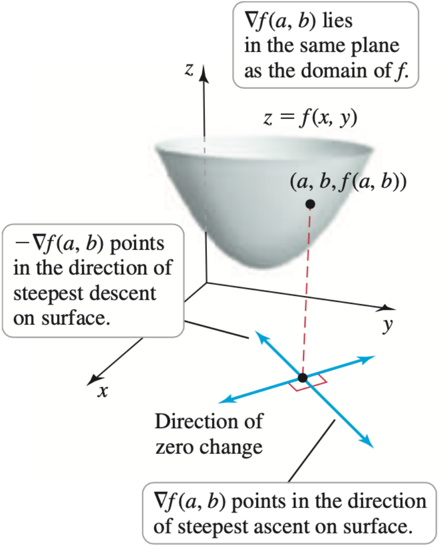
$$D_{\mathbf{u}} f(x,y) = |\nabla f(x,y)| \cos(\theta) \le |\nabla f(x,y)|$$

Rmk: $\cos(\theta)=0$ if $\theta = \frac{\pi}{2}$,
i.e $u \perp \nabla f$

Information given by the gradient

1. $|\nabla f(x,y)|$ is the maximal possible directional derivative
2. The direction $\mathbf{u} = \frac{\nabla f(x,y)}{|\nabla f(x,y)|}$ is the one of maximal ascent
3. The direction $\mathbf{u} = -\frac{\nabla f(x,y)}{|\nabla f(x,y)|}$ is the one of maximal desccent
4. If $\mathbf{u} \perp \nabla f(x,y)$, the directional derivative is 0

Interpretation of gradient: illustration



Example of steepest descent (1)

Function:

$$f(x, y) = 4 + x^2 + 3y^2$$

$\frac{35}{4} = f(2, -\frac{1}{2})$

Questions:

1. If you are located on the paraboloid at the point $(2, -\frac{1}{2}, \frac{35}{4})$
   $\hookrightarrow$ In which direction should you move in order to ascend on the surface at the maximum rate?
2. If you are located on the paraboloid at the point $(2, -\frac{1}{2}, \frac{35}{4})$
   $\hookrightarrow$ In which direction should you move in order to descend on the surface at the maximum rate?
3. At the point (3, 1, 16), in what direction(s) is there no change in the function values?

Function $f(x,y) = 4 + x^2 + 3y^2$, point: $(2, -\frac{1}{2})$

Maximal ascent : in the direction of $\nabla f(x,y)$

Here $\nabla f(x,y) = \langle 2x, 6y \rangle$

Thus $\nabla f(2, -\frac{1}{2}) = \langle 4, -3 \rangle$

Direction of max ascent

$$u = \frac{\nabla f(2,-\frac{1}{2})}{|\nabla f(2,-\frac{1}{2})|} = \frac{\langle 4, -3 \rangle}{(4^2+3^2)^{1/2}}$$

$$\boxed{u = \langle 4/5, -3/5 \rangle}$$

→ growth rate $= |\nabla f(2,-\frac{1}{2})|$ $= 5$

Direction of max descent

$$v = \frac{-\nabla f(2,-\frac{1}{2})}{|\nabla f(2,-\frac{1}{2})|} = \langle -4/5, 3/5 \rangle$$

↳ rate $= -5$

Example of steepest descent (2)

Gradient:

$$\nabla f(x, y) = \langle 2x,\ 6y \rangle$$

Thus

$$\nabla f\left(2, -\frac{1}{2}\right) = \langle 4,\ -3 \rangle$$

Steepest ascent direction: We get

$$\mathbf{u} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle,$$

with rate of ascent

$$\left|\nabla f\left(2, -\frac{1}{2}\right)\right| = 5$$

Example of steepest descent (3)

Steepest descent direction: We get

$$\mathbf{v} = -\mathbf{u} = \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle,$$

with rate of descent

$$-\left| \nabla f \left(2, -\frac{1}{2} \right) \right| = -5$$