Use of the gradient in $\mathbb{R}^2$

Function $f(x,y) = z$. Then

(i) $\nabla f(x,y)$ gives direction of max ascent

(ii) $-\nabla f(x,y)$ " " " " descent

(iii) $\perp \nabla f(x,y)$ gives tangent to level curve

Use of the gradient in $\mathbb{R}^3$

For an implicit function $F(x,y,z) = 0$, tangent plane is given by

$$\nabla F(a,b,c) \cdot \langle x-a,\ y-b,\ z-c \rangle = 0$$

Tangent plane for $z = f(x, y)$ (explicit function)

S_1: $\cos(z) + e^{xz - 2yx} = 0$ (implicit)

S_2: $z = x^2 + y^2$ (explicit)

Definition 10.

Let $f(x, y)$ be such that

- f differentiable at (a, b)
- S is the surface $z = f(x, y)$

Then the tangent plane to S at $(a, b, f(a, b))$ is given by

$$z = f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b) + f(a, b)$$

Otherwise stated

$$z - f(a,b) = \nabla f(a,b) \cdot \langle x-a, y-b \rangle$$

same equation

$\nabla f(a,b)$ ↳ 2d-gradient

Example of tangent plane for $z = f(x, y)$ (1)

Surface: Paraboloid of the form

$$z = f(x, y) = 32 - 3x^2 - 4y^2$$

Question:

- Tangent plane at $(2, 1, 16)$

Function $z = f(x,y) = 32 - 3x^2 - 4y^2$

Point $(a,b) = (2,1)$ Then $z = 32 - 3\times 2^2 - 4\times 1^2 = 16$

Partial derivatives

$f_x(x,y) = -6x$ $f_y(x,y) = -8y$

Tangent plane at $(a,b) = (2,1)$

$z = 16 + (-6\times 2)(x-2) + (-8\times 1)(y-1)$

$$\boxed{z = -12x - 8y + 48}$$

Example of tangent plane for $z = f(x, y)$ (2)

Partial derivatives: We have

$$f_x = 6x, \qquad f_y = -8y$$

Thus

$$f_x(2,1) = -12, \qquad f_y(2,1) = -8$$

Tangent plane:

$$z = -12x - 8y + 48$$

Linear approx for functions of 1 variable (Repeat)

Situation: We have

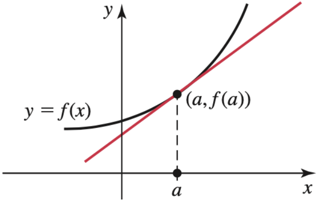
- $y = f(x)$

Tangent vector at a:

$$\mathbf{t} = (1, f'(a))$$

Linear approximation: Near a we have

$$f(x) \simeq f(a) + f'(a)(x - a)$$



Linear approximation for functions of 2 variables

Definition 11.

Let $f(x, y)$ be such that

- f differentiable at (a, b)
- S is the surface $z = f(x, y)$

Then the linear approximation to S at $(a, b, f(a, b))$ is given by

$$L(x, y) = f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b) + f(a, b)$$

Remark: Another popular form of the linear approximation is

$$\Delta z \simeq f_x \mathrm{d}x + f_y \mathrm{d}y$$

$f(x,y) - f(a,b)$

Example of infinitesimal change (1)

Function:

$$z = f(x, y) = x^2 y$$

Question: Evaluate the percentage of change in z if

- x is increased by 1%
- y is decreased by 3%

Function $z = f(x,y) = x^2 y$

Rate of change. According to Def II,

$$\Delta z = f_x \, dx + f_y \, dy$$

$$\Delta z = 2xy \, dx + x^2 \, dy$$

Thus **% of change** is given by

$$\frac{\Delta z}{z} \simeq \frac{2xy}{z} dx + \frac{x^2}{z} dy \qquad z = x^2 y$$

$$= \frac{2xy}{x^2 y} dx + \frac{x^2}{x^2 y} dy$$

$$\frac{\Delta z}{z} \simeq 2 \frac{dx}{x} + \frac{dy}{y}$$

Application If $x \nearrow$ by 1%, $y \searrow$ by 3%,
then $\Delta z / z = 2 - 3 = \boxed{-1\%}$

Example of infinitesimal change (2)

Small change in z:

$$\mathrm{d}z \simeq f_x \mathrm{d}x + f_y \mathrm{d}y = 2xy\mathrm{d}x + x^2\mathrm{d}y$$

Small percentage change in z:

$$\frac{\mathrm{d}z}{z} = \frac{2xy}{z}\mathrm{d}x + \frac{x^2}{z}\mathrm{d}y = \frac{2}{x}\mathrm{d}x + \frac{1}{y}\mathrm{d}y$$

If $\frac{\mathrm{d}x}{x} = .01$ and $\frac{\mathrm{d}y}{y} = -.03$:

$$\frac{\mathrm{d}z}{z} = -.01 = -1\%$$

Outline

Max and min for functions of 1 variable

Situation: We have

- $y = f(x)$

Critical point: $(c, f(c))$ whenever

$$f'(c) = 0$$

Second derivative test: If $(c, f(c))$ is critical then

1. If $f''(c) > 0$, there is a local minimum
2. If $f''(c) < 0$, there is a local maximum
3. If $f''(c) = 0$, the test is inconclusive

In terms of Taylor exp, close to c (critical)

$$f(x) - f(c) \simeq \frac{1}{2} f''(c) (x-c)^2$$

If $f''(c) > 0$, $f(x) - f(c) > 0$, then local min at c

Critical points for functions of 2 variables

Definition 12.

Let

- f function of 2 variables
- (a, b) interior point in the domain of f

Then (a, b) is a critical point of f if

$$f_x(a, b) = 0, \quad \text{and} \quad f_y(a, b) = 0,$$

or if one of the partial derivatives f_x, f_y does not exist at (a, b)

Otherwise stated $\nabla f(a,b) = \vec{0}$

Second derivative test

Theorem 13.

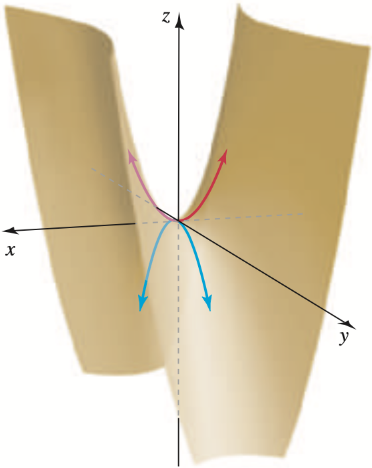
For f twice diff. function, define the discriminant of f as

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2$$

Then for a critical point (a,b) the following holds true:

1. If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, we have a local max
2. If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, we have a local min
3. If $D(a,b) < 0$, we have a saddle point
4. If $D(a,b) = 0$, the test is inconclusive

Saddle point for an hyperboloid



The hyperbolic paraboloid $z = x^2 - y^2$ has a saddle point at $(0, 0)$.

Hyperboloids in architecture

Hyperboloids in the food industry

Example of critical points analysis (1)

Function:

$$f(x, y) = x^2 + 2y^2 - 4x + 4y + 6$$

Problem:

Use second derivative test to classify the critical points of f

Function $f(x,y) = x^2 + 2y^2 - 4x + 4y + 6$

Gradient

$f_x(x,y) = 2x - 4$

$f_y(x,y) = 4y + 4$

Critical point $\begin{cases} 2x - 4 = 0 \\ 4y + 4 = 0 \end{cases}$

we get $(2, -1)$

At $(2,-1)$, second derivatives are

$f_{xx} = 2 \quad f_{yy} = 4 \quad f_{xy} = f_{yx} = 0$

Thus $D(x,y) = 8 > 0, \; f_{xx} > 0$ $\Rightarrow$ loc min

Example of critical points analysis (2)

Partial derivatives:

$$f_x = 2x - 4, \qquad f_y = 4y + 4$$

Critical point:

$$(2, -1)$$

Critical value of f:

$$f(2, -1) = 0$$

Example of critical points analysis (3)

Second derivatives:

$$f_{xx} = 2, \qquad f_{xy} = f_{yx} = 0, \qquad f_{yy} = 4$$

Discriminant:

$$D(x, y) = 8 > 0$$

Second derivative test: We have

$$D(2, -1) > 0, \; f_{xx}(2, -1) > 0 \quad \Longrightarrow \quad \text{Local minimum at } (2, -1)$$