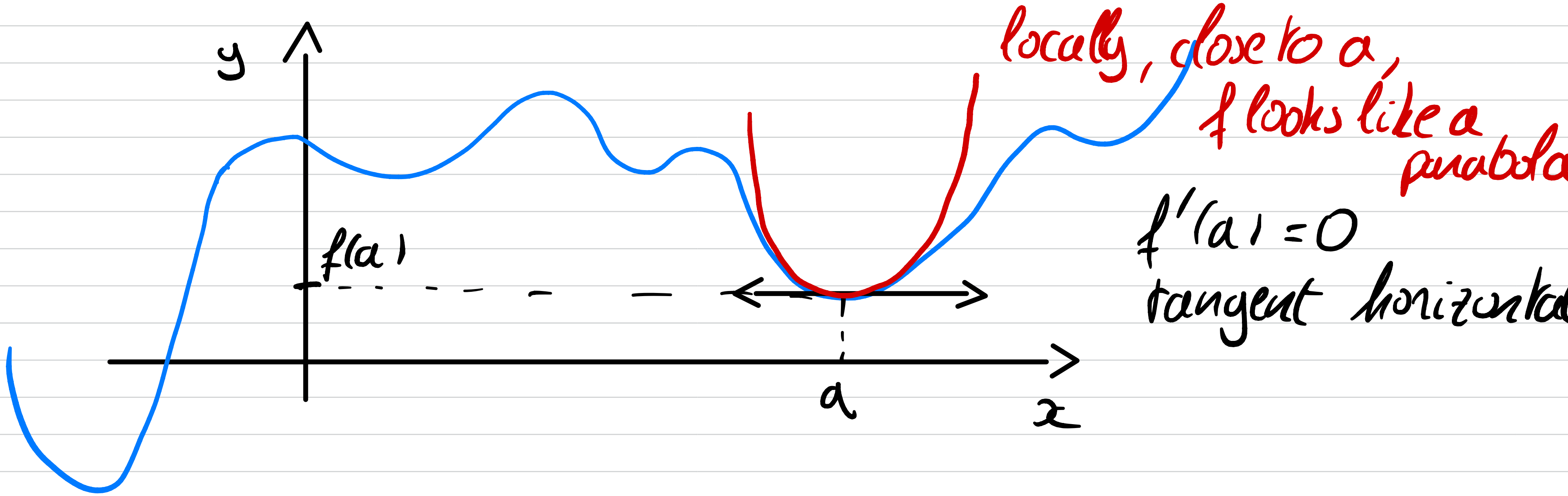
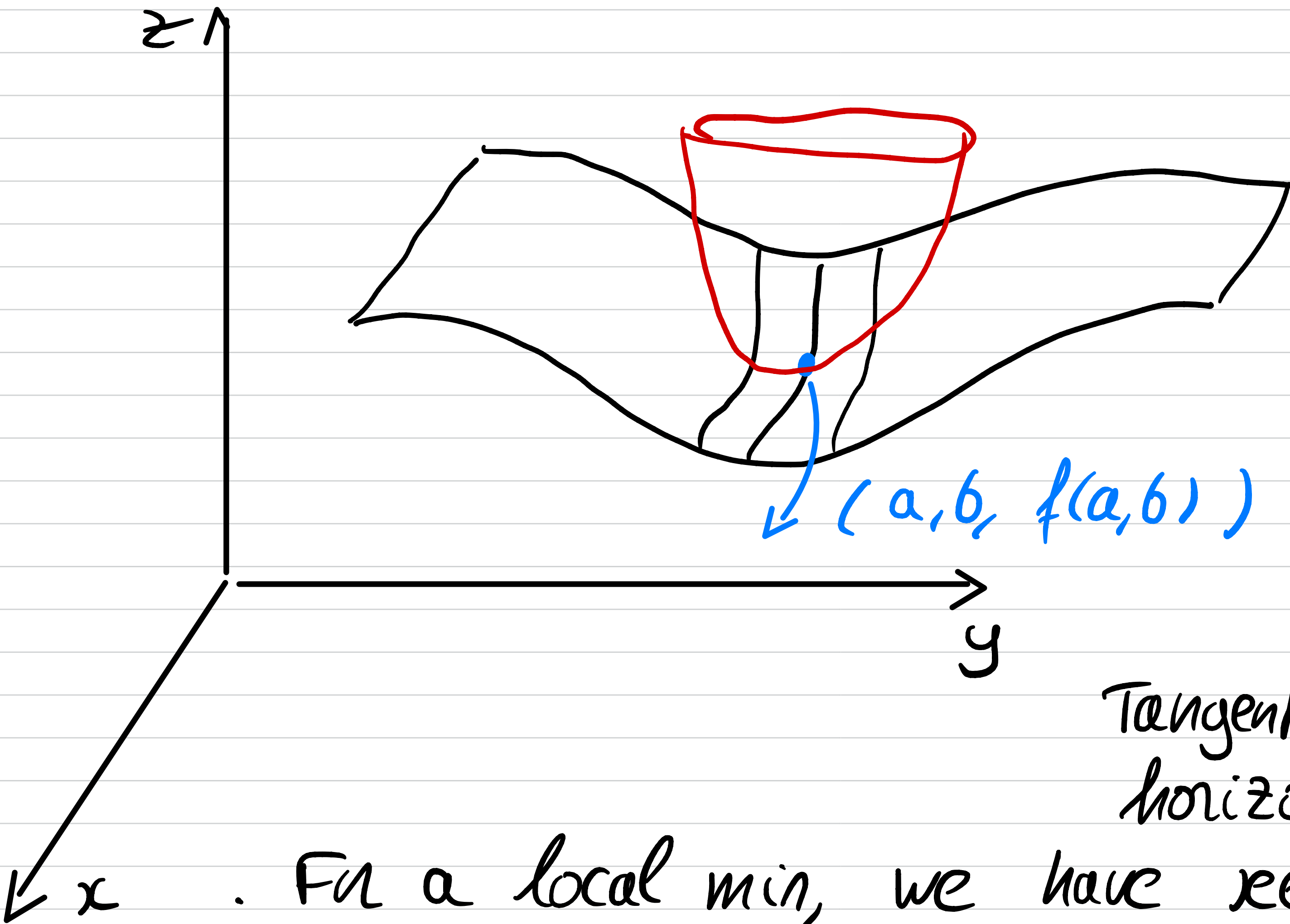
Local Minimums for $y = f(x)$



$f'(a) = 0$

tangent horizontal

Local minimum for $f(x,y) = z$



Rmk. For a max, flip everything upside down

- Other situation: saddle point

- We have $\nabla f(a,b) = 0$

Tangent plane is horizontal

- For a local min, we have seen $f_{xx}(a,b) > 0$ and $D(a,b) = f_{xx} f_{yy} - (f_{xy})^2 > 0$

$\Rightarrow$ locally, f look like a paraboloid

Example of critical points analysis (1)

paraboloid

Function:

$$f(x, y) = x^2 + 2y^2 - 4x + 4y + 6$$

Problem:

Use second derivative test to classify the critical points of f

Function $f(x,y) = x^2 + 2y^2 - 4x + 4y + 6$

Gradient

$$f_x(x,y) = 2x - 4$$

$$f_y(x,y) = 4y + 4$$

Critical point $\begin{cases} 2x - 4 = 0 \\ 4y + 4 = 0 \end{cases}$

we get $(2, -1)$ and $f(2,-1) = 0$

At $(2,-1)$, second derivatives are

$f_{xx} = 2$ $\quad f_{yy} = 4$ $\quad f_{xy} = f_{yx} = 0$

Thus $D(x,y) = 8 > 0$, $f_{xx} > 0$ $\Rightarrow$ loc min

Example of critical points analysis (2)

Partial derivatives:

$$f_x = 2x - 4, \qquad f_y = 4y + 4$$

Critical point:

$$(2, -1)$$

Critical value of f:

$$f(2, -1) = 0$$

Example of critical points analysis (3)

Second derivatives:

$$f_{xx} = 2, \qquad f_{xy} = f_{yx} = 0, \qquad f_{yy} = 4$$

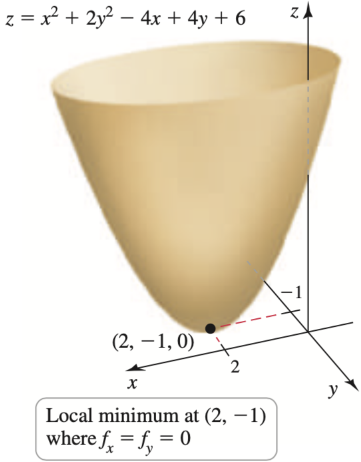
Discriminant:

$$D(x, y) = 8 > 0$$

Second derivative test: We have

$$D(2, -1) > 0, \; f_{xx}(2, -1) > 0 \quad \Longrightarrow \quad \text{Local minimum at } (2, -1)$$

Example of critical points analysis (4)



Second example (1)

Function:

$$f(x, y) = xy(x-2)(y+3)$$

Problem:

Use second derivative test to classify the critical points of f

Function $f(x,y) = xy(x-2)(y+3)$

① **Gradient**

$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

$$\nabla f = \langle (2x-2)y(y+3), (2y+3)x(x-2) \rangle$$

② **Critical points**. Where $\nabla f(x,y) = \vec{0}$ we get

$(1, -3/2)$ $(0, -3)$

$(0, 0)$ $(2, -3)$

$(2, 0)$

Rmk Horizontal tangent plane for all those points.

(3) For each critical point we should compute f_{xx} and $D(x,y)$

$$\nabla f = \langle (2x-2)y(y+3),\ (2y+3)x(x-2) \rangle$$

we have

$$f_{xx}(x,y) = 2y(y+3) \qquad f_{yy}(x,y) = 2x(x-2)$$

$$f_{xy}(x,y) = (2x-2)(2y+3)$$

$$\boxed{D(x,y) = f_{xx} f_{yy} - (f_{xy})^2}$$

Second example (2)

Partial derivatives:

$$f_x = 2y(x-1)(y+3), \qquad f_y = x(x-2)(2y+3)$$

Critical points:

$$(0,0), \quad (2,0), \quad (1,-\frac{3}{2}), \quad (0,-3), \quad (2,-3),$$

Second example (3)

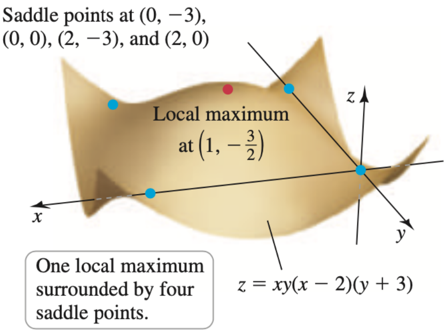
Second derivatives:

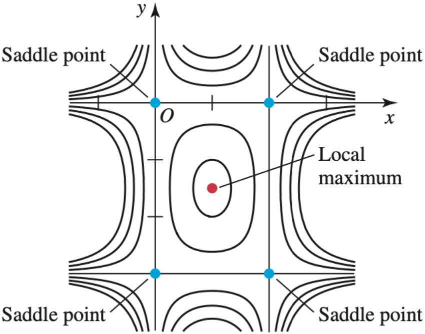
$$f_{xx} = 2y(y+3), \quad f_{xy} = 2(2y+3)(x-1), \qquad f_{yy} = 2x(x-2)$$

Analysis of critical points:

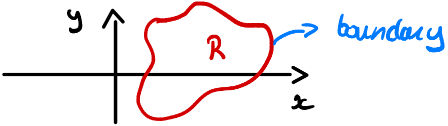
(x, y)	$D(x, y)$	f_{xx}	**Conclusion**
$(0, 0)$	-36	0	Saddle point
$(2, 0)$	-36	0	Saddle point
$(1, -3/2)$	9	$-9/2$	Local maximum
$(0, -3)$	-36	0	Saddle point
$(2, -3)$	-36	0	Saddle point

Second example (4)





Absolute maximum



Proposition 14.

Let

- f continuous function of 2 variables
- R closed region of $\mathbb{R}^2$

In order to find the maximum of f in R, we proceed as follows:

1. Determine the values of f at all critical points in R.
2. Find the maximum and minimum values of f on the boundary of R.
3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on R.

Example of global maximum (1)

Function:

$$z = f(x, y) = x^2 + y^2 - 2x - 4y$$

Region:

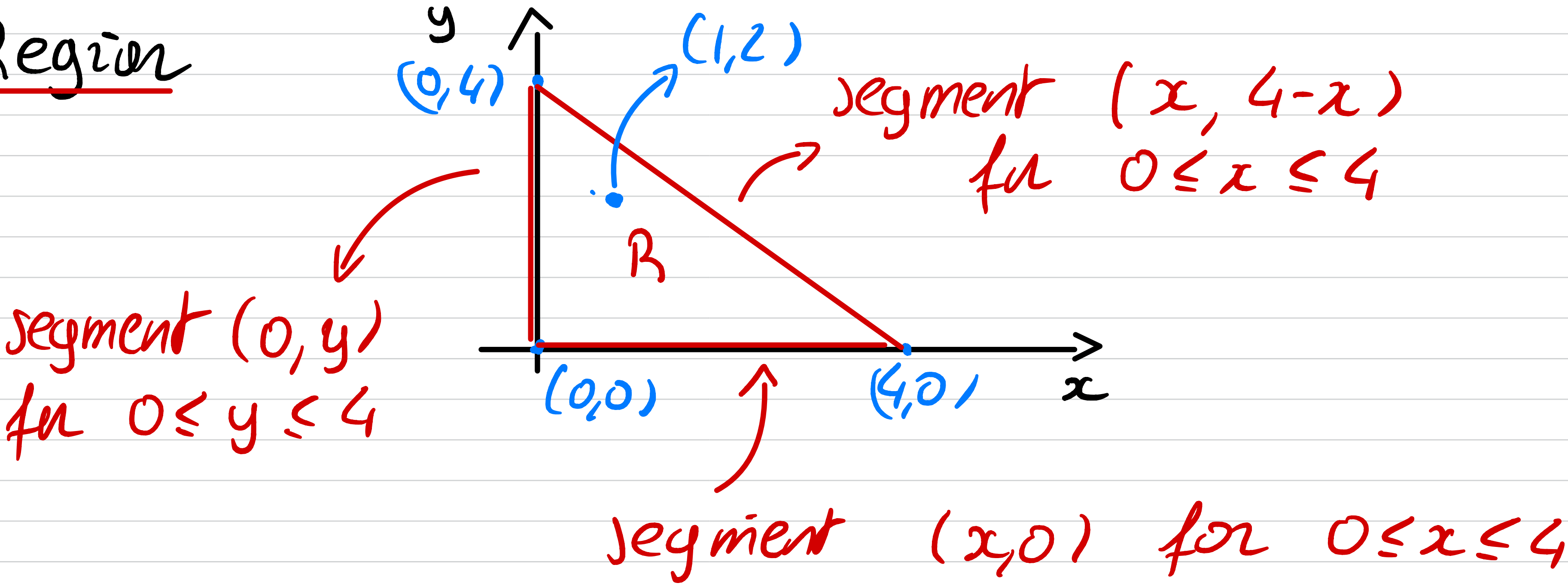
$$R = \{(x, y);\ (x, y) \text{ within triangle with vertices } (0, 0), (0, 4), (4, 0)\}$$

Question:

Find global maximum of f on region R

Function $z = f(x,y) = x^2 + y^2 - 2x - 4y$

Region



Function $z = f(x,y) = x^2 + y^2 - 2x - 4y$

① critical points inside R

$\nabla f(x,y) = \langle 2x-2, \ 2y-4 \rangle$

we get one critical point

$(1,2)$

we have $0 \leq 1 \leq 4$, $0 \leq 2 \leq 4$

and $4-1=3 \geq 2$

Thus $(1,2) \in R$ (critical point inside R)

$f(x,y) = x^2+y^2-2x-4y$

② -(a) Boundary $0 \le x \le 4, \quad y=0$

Define $f(x,0) = x^2-2x \equiv h(x)$

Compute $h'(x) = 2x-2$

↳ 1-d critical point: $x=1$

We thus get 3 points of interest:

$(0,0)$	$(1,0)$	$(4,0)$

② -(b) Boundary $0 \le x \le 4, \quad y = 4-x$

$f(x,4-x) = x^2+(4-x)^2-2x-4(4-x)$

$= \dots = 2x^2-6x \equiv k(x)$

$k'(x) = 4x-6$.

$4-\frac{3}{2}$

Point of interest $\left(\frac{3}{2}, \frac{5}{2}\right)$, $(0,4)$

(2) -(c)

Boundary $0 \le y \le 4, x = 0$

⋮

Points of interest	Value of f
$(1,2)$	$-5 \longrightarrow$ min
$(0,0)$	0
$(1,0)$	-1
$(4,0)$	$8 \longrightarrow$ max
$(\frac{3}{2},\frac{5}{2})$	$-9/2$
$(0,4)$	0
$\vdots$	$\vdots$

Example of global maximum (2)

Partial derivatives:

$$f_x = 2x - 2, \qquad f_y = 2y - 4$$

Critical point:

$$(1,2), \quad \text{with} \quad f(1,2) = -5$$

Example of global maximum (3)

Boundary 1: On $y = 0$, $0 \leq x \leq 4$ we have

$$f(x, y) = x^2 - 2x \equiv g(x), \qquad g'(x) = 2(x - 1)$$

Points of interest on boundary 1: We get

$$(0, 0), \qquad (1, 0), \qquad (0, 4)$$

and

$$f(0, 0) = 0, \qquad f(1, 0) = -1, \qquad f(4, 0) = 8$$

Example of global maximum (4)

Boundary 2: On $y = 4 - x$, $0 \leq x \leq 4$ we have

$$f(x, y) = 2x^2 - 6x \equiv h(x), \qquad h'(x) = 4x - 6$$

Points of interest on boundary 2: We get

$$(0, 4), \qquad \left(\frac{3}{2}, \frac{5}{2}\right), \qquad (4, 0)$$

and

$$f(0, 4) = 0, \qquad f\left(\frac{3}{2}, \frac{5}{2}\right) = -\frac{9}{2}, \qquad f(4, 0) = 8$$

Example of global maximum (5)

Boundary 3: On $x = 0$, $0 \leq y \leq 4$ we have

$$f(x, y) = y^2 - 4y \equiv k(y), \qquad k'(y) = 2(y - 2)$$

Points of interest on boundary 3: We get

$$(0, 0), \qquad (0, 2), \qquad (0, 4)$$

and

$$f(0, 0) = 0, \qquad f(0, 2) = -4, \qquad f(0, 4) = 0$$

Example of global maximum (6)

Summary of points of interest:

$$f(0,0) = 0, \qquad f(1,0) = -1, \qquad f(4,0) = 8$$

$$f(0,4) = 0, \qquad f\left(\frac{3}{2}, \frac{5}{2}\right) = -\frac{9}{2}, \qquad f(4,0) = 8$$

$$f(0,0) = 0, \qquad f(0,2) = -4, \qquad f(0,4) = 0, \qquad f(1,2) = -5$$

Absolute minimum: at $(1,2)$ and

$$f(1,2) = -5$$

Absolute maximum: at $(4,0)$ and

$$f(4,0) = 8$$