For functions of {2 variables / 1 variable}

{Differentiation / Gradient} → tangent, slopes ↗ tangent plane

approximations (Taylor) (1st order)

max-min problems ↪ for fct of 2 variables

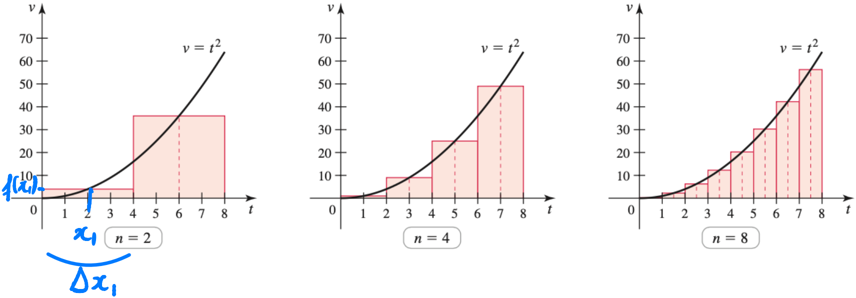
Integration → {area / volume} under {curve / surface}

Outline

Integration in dimension 1 (1)

Approximation procedure:

Area under a curve is approximated by sum of rectangle areas

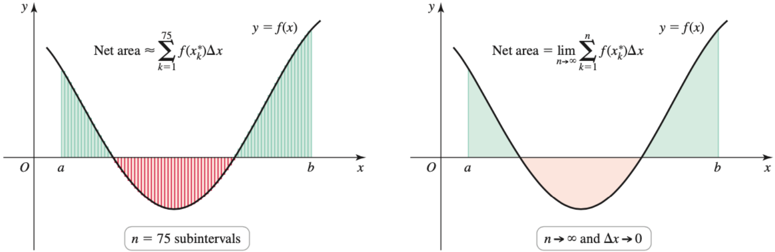


area of rectangle: $f(x_1)\Delta x_1$

Integration in dimension 1 (2)

Riemann integral: In the limit we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x_k = \int_a^b f(x)\,\mathrm{d}x$$



Volume approximation (1)

Aim: Approximate the volume V
$\hookrightarrow$ Under the surface defined by f on rectangle $R = [a, b] \times [c, d]$

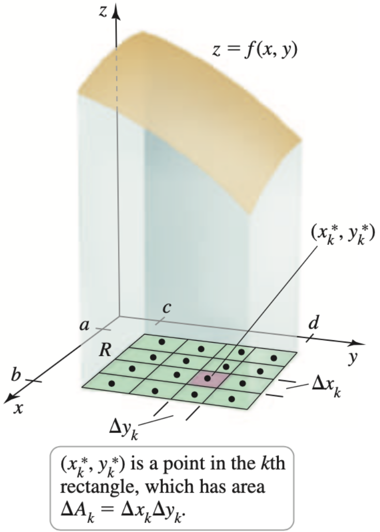
Approximation:

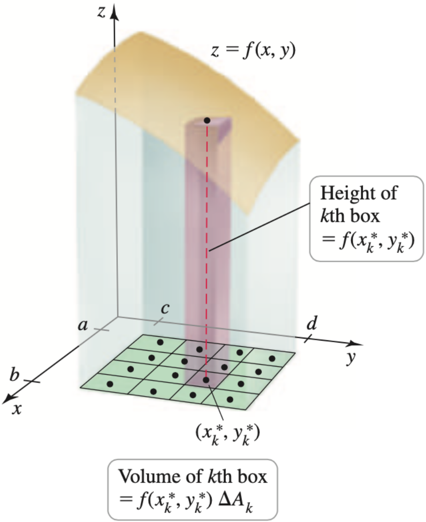
- Divide R into boxes centered at (x_k^*, y_k^*)
- Area of each box: $\Delta A_k = \Delta x_k \Delta y_k$

Then the volume is approximated as

$$V \simeq \sum_{k=1}^{n} f(x_k^*, y_k^*) \, \Delta A_k$$

Volume approximation (2)

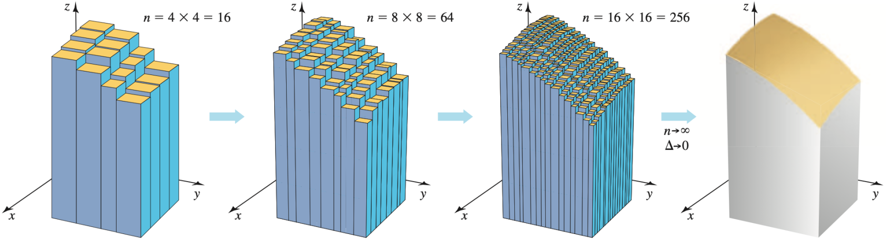




Integration in dimension 1 (3)

Double integral: In the limit we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k = \int\int_R f(x,y)\, \overbrace{\mathrm{d}A}^{= dx\, dy}$$



Computing double integrals

integrate inside first

then outside

Basic recipe:

$$\int\left(\int f(x,y)\,dx\right)dy = \int\left(\int f(x,y)\,dy\right)dx$$

1. Integrate inside out
2. While integrating wrt one variable, keep the other one constant
3. Fubini: The order of integration does not matter

Example of double integration (1)

Function:

$$z = f(x, y) = 6 - 2x - y$$

Region: Rectangle

$$R = [0, 1] \times [0, 2]$$

Problem: Compute

$$\int\int_R f(x, y)\,\mathrm{d}A$$

Function $f(x,y) = 6 - 2x - y$

Rectangle $B = [0,1] \times [0,2]$

Integral

$$\int_0^1 \left(\int_0^2 (6 - 2x - y)\, dy \right) dx$$

$$= 6y - 2xy - \frac{y^2}{2} \Big|_0^2 = 12 - 4x - 2$$

$$= 10 - 4x$$

$$= \int_0^1 (10 - 4x)\, dx$$

$$= 10x - 2x^2 \Big|_0^1$$

$$= 10 - 2 = 8$$

Function $f(x,y) = 6 - 2x - y$

Rectangle $B = [0,1] \times [0,2]$

Let us check that the order of integration does not matter

$$\int_0^2 \left(\int_0^1 (6 - 2x - y)\, dx \right) dy$$

$$= 6x - x^2 - yx \Big|_0^1 = 6 - 1 - y = 5 - y$$

$$= \int_0^2 (5 - y)\, dy = 5y - \frac{y^2}{2} \Big|_0^2$$

$$= 10 - 2 = 8$$ (same result as before)

Example of double integration (2)

Integrating: We get

$$\int\int_R f(x,y)\,\mathrm{d}A = \int_0^1 \left(\int_0^2 (6-2x-y)\,\mathrm{d}y\right)\mathrm{d}x$$
$$= \int_0^1 (10-4x)\,\mathrm{d}x$$
$$= 10x - 2x^2\Big|_0^1$$

Area: We get

$$\int\int_R f(x,y)\,\mathrm{d}A = 8$$

To be checked: We also have

$$\int\int_R f(x,y)\,\mathrm{d}A = \int_0^2 \left(\int_0^1 (6-2x-y)\,\mathrm{d}x\right)\mathrm{d}y$$

Illustration: integrating first in y

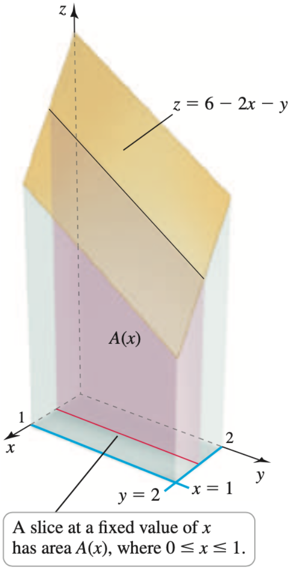
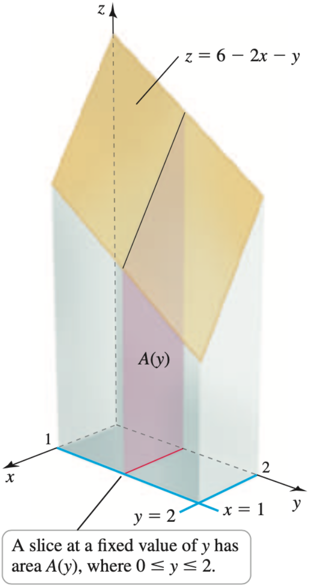


Illustration: integrating first in x



Choosing the correct order of integration (1)

Function:

$$z = f(x, y) = y^5 x^2 e^{x^3 y^3}$$

Region: Rectangle

$$R = [0, 2] \times [0, 1]$$

Problem: Compute

$$\int\int_R f(x, y)\, \mathrm{d}A$$

$R = \overbrace{[0,2]}^{x} \times \overbrace{[0,1]}^{y}$

Function $f(x,y) = y^5 x^2 e^{x^3 y^3}$

Then $\int_0^1 f(x,y)\,dy = x^2 \int_0^1 y^5 e^{x^3 y^3}\,dy$

we would like to use substitution:

$$\int u' e^u = e^u \quad (+c)$$

Here this does not work for $\int_0^1 y^5 e^{x^3 y^3} dy$ ($y^5 \neq (y^3)'$)

However if we consider x first:

$$\int_0^2 f(x,y)\,dx = y^5 \int x^2 e^{y^3 x^3}\,dx$$

$$= \frac{y^2}{3} \int \overbrace{y^3 \times 3x^2}^{u'} e^{y^3 x^3}\,dx \quad (u = y^3 x^3)$$

$$= \frac{y^2}{3} e^{y^3 x^3} \Big|_0^2 = \frac{y^2}{3}\left(e^{8y^3} - 1\right)$$

2-d integral

$$\int_0^1 \left(\int_0^2 f(x,y)\, dx \right) dy$$

$$= \int_0^1 \frac{y^2}{3} \left(e^{8y^3} - 1 \right) dy$$

$\vdots$

$$= \frac{1}{72} e^8 - \frac{1}{9} \simeq 41.\ldots$$

Choosing the correct order of integration (2)

Order of integration: We integrate wrt x first and compute

$$\begin{aligned}
\int\int_R f(x,y)\,\mathrm{d}A &= \int_0^1 y^2 \left(\int_0^2 y^3 x^2 e^{x^3 y^3}\,\mathrm{d}x\right)\mathrm{d}y \\
&= \frac{1}{3}\int_0^1 y^2 \left(e^{x^3 y^3}\Big|_{x=0}^{x=2}\right)\mathrm{d}y \\
&= \frac{1}{3}\int_0^1 y^2 \left(e^{8y^3} - 1\right)\mathrm{d}y \\
&= \frac{1}{72}e^8 - \frac{1}{8} \\
&\simeq 71.28
\end{aligned}$$

Average value

<u>1-d case</u> : $\bar{f} = \frac{1}{b-a} \int_a^b f(x)\,dx$

Definition 1.

Let

- f function of 2 variables
- R rectangle

Then the average value of f on R is given by

$$\bar{f} = \frac{1}{\text{Area}(R)} \int\int_R f(x,y)\,\mathrm{d}A$$

Example of average value (1)

Function:

$$z = f(x, y) = 2 - x - y$$

Region: Rectangle

$$R = [0, 2] \times [0, 2]$$

Problem:
Compute the average value of f on R

<u>Function</u> $2 - x - y$

$R = [0,2] \times [0,2]$

Then

$$\bar{f} = \frac{1}{2\times 2} \int_0^2 \left(\int_0^2 (2-x-y)\, dx \right) dy$$

$$= 2x - \frac{x^2}{2} - yx \Big|_0^2$$

$\vdots$

$$\boxed{\bar{f} = 0}$$

(f centered)

Example of average value (2)

Integrating: We get

$$\begin{aligned}\bar{f} &= \frac{1}{\text{Area}(R)} \int\int_R f(x,y)\,\mathrm{d}A \\ &= \frac{1}{4}\int_0^2 \left(\int_0^2 (2-x-y)\,\mathrm{d}x\right)\mathrm{d}y \\ &= \frac{1}{4}\int_0^2 (2-2y)\,\mathrm{d}y \\ &= 0\end{aligned}$$

Average value: We find that f is centered on R, ie

$$\bar{f} = 0$$