Outline

1 Double integrals over rectangular regions

2 Double integrals over general regions

3 Double integrals in polar coordinates

4 Triple integrals

5 Triple integrals in cylindrical and spherical coordinates

6 Integrals for mass calculations

Last time: Integrals over rectangles, like

$$\boxed{I = \int_0^1 \left(\int_1^2 f(x,y)\, dy \right) dx}$$

- Integrate inside-out
- Order does not matter:

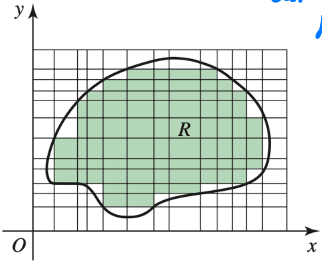
$$I = \int_1^2 \left(\int_0^1 f(x,y)\, dx \right) dy$$

Description of the problem

New situation:
The region R of integration is not a rectangle

Consequence: Order of integration is important
$\hookrightarrow$ and cannot be switched arbitrarily

R remains the same,
but we have to change
the description of R



Special form of domain

Particular case: We have

element of

$$R = \Big\{(x,y) \in \mathbb{R}^2;\ \underbrace{x \in [a,b]}_{\text{then integrate wrt } x},\ \underbrace{f(x) \le y \le g(x)}_{\text{first integrate wrt } y}\Big\}$$

Recipe:

Integrate wrt variable with constant bounds last

Example of integration (1)

Function:

$$z = f(x,y) = xy^2$$

Region: Of the form

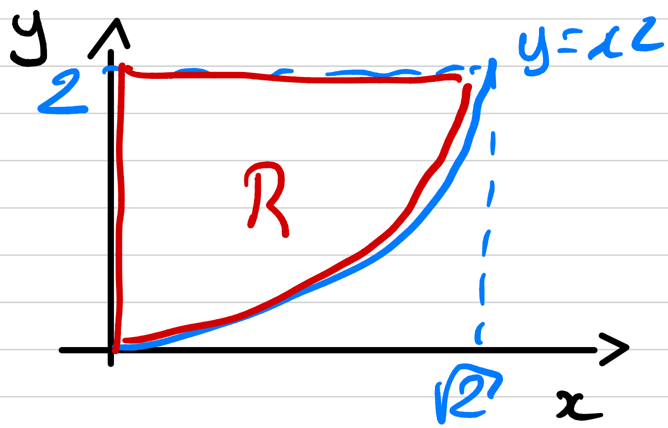
$$R = \left\{(x,y) \in \mathbb{R}^2;\ x \in [0, \sqrt{2}],\ x^2 \le y \le 2\right\}$$

Problem: Compute

$$\int\int_R f(x,y)\,\mathrm{d}A$$

Function $\quad f(x,y) = xy^2$

Region $\quad R = \{(x,y) \in \mathbb{R}^2 \,;\ x \in [0,\sqrt{2}] \,,\ x^2 \le y \le 2\}$



$$I = \int_0^{\sqrt{2}} \underbrace{\left(\int_{x^2}^{2} xy^2 \, dy \right)} dx$$

$$= \frac{1}{3} x y^3 \Big|_{x^2}^{2}$$

$$= \frac{8}{3} x - \frac{1}{3} x^7$$

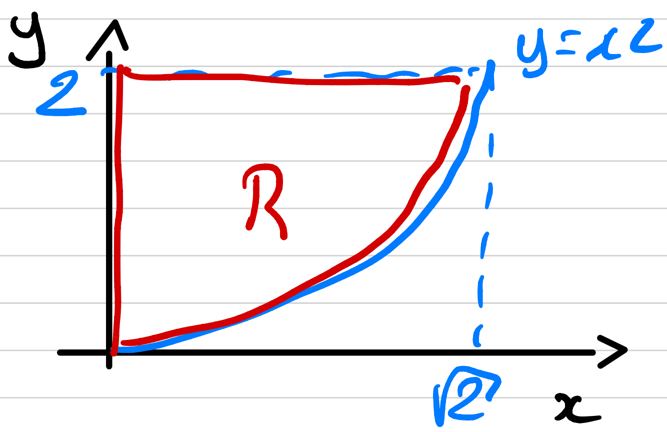
Thus

$$I = \int_0^{\sqrt{2}} \left(\frac{8}{3} x - \frac{1}{3} x^7 \right) dx$$

$$= \frac{4}{3} x^2 - \frac{1}{24} x^8 \Big|_0^{\sqrt{2}} \quad \Rightarrow \quad \boxed{I = 2}$$

Function $\quad f(x,y) = xy^2$

Region $\quad R = \{(x,y) \in \mathbb{R}^2 ;\ x \in [0, \sqrt{2}],\ x^2 \le y \le 2\}$ $\quad \Leftrightarrow x \le \sqrt{y}$



Aim: switch order of integration for x and y

Main issue: change description of R

Here we get

$$R = \{(x,y) \in \mathbb{R}^2 ;\ y \in [0,2],\ 0 \le x \le \sqrt{y}\}$$

we obtain

$$I = \int_0^2 \underbrace{\left(\int_0^{\sqrt{y}} xy^2 \, dx \right)}_{= \frac{1}{2} x^2 y^2 \Big|_0^{\sqrt{y}} = \frac{1}{2} y^3 - 0} dy$$

Further computations

$$I = \int_0^2 \underbrace{\left(\int_0^{\sqrt{y}} x y^2 \, dx \right)}_{= \frac{1}{2} x^2 y^2 \Big|_0^{\sqrt{y}} = \frac{1}{2} y^3 - 0} dy$$

$$= \frac{1}{2} \int_0^2 y^3 \, dy = \frac{1}{2} \times \frac{y^4}{4} \Big|_0^2$$

$$= \frac{16}{8} = 2$$

$$\boxed{I = 2 \text{ , as before}}$$

Example of integration (2)

Order of integration: We integrate wrt y first and compute

$$\begin{aligned}\int\int_R f(x,y)\,\mathrm{d}A &= \int_0^{\sqrt{2}} \left(\int_{x^2}^{2} xy^2\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_0^{\sqrt{2}} \frac{8}{3}x - \frac{1}{3}x^7\,\mathrm{d}x \\ &= \frac{8}{6}x^2 - \frac{1}{24}x^8\Big|_0^{\sqrt{2}} \\ &= 2\end{aligned}$$

Example of integration (3)

Switching order of integration:

One has to be more careful than for rectangles. We get that

$$R = \left\{(x, y) \in \mathbb{R}^2;\ x \in [0, \sqrt{2}],\ x^2 \le y \le 2\right\}$$

can also be written as

$$R = \left\{(x, y) \in \mathbb{R}^2;\ y \in [0, 2],\ 0 \le x \le \sqrt{y}\right\}$$

Example of integration (4)

Integration with order switched: We integrate wrt x first and compute

$$\begin{aligned}
\int\int_R f(x,y)\,\mathrm{d}A &= \int_0^2 \left(\int_0^{\sqrt{y}} xy^2\,\mathrm{d}x \right) \mathrm{d}y \\
&= \frac{1}{2}\int_0^2 y^3\,\mathrm{d}y \\
&= \frac{1}{8}\,y^4\Big|_0^2 \\
&= 2
\end{aligned}$$

Switching order of integration (1)

Function: consider a general function

$$z = f(x, y)$$

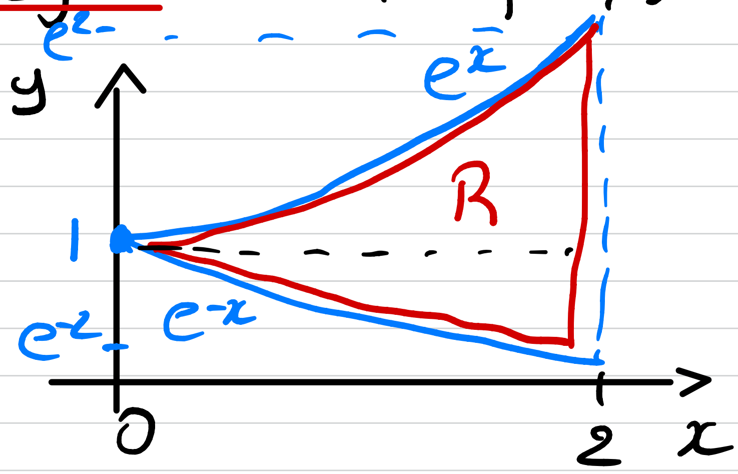
Region: Of the form

$$R = \left\{(x, y) \in \mathbb{R}^2;\, x \in [0, 2],\, \underbrace{e^{-x}}_{f(x)} \le y \le \underbrace{e^{x}}_{g(x)}\right\}$$

Problem: Switch the order of integration for

$$\int\int_R f(x, y)\,\mathrm{d}A = \int_0^2 \int_{e^{-x}}^{e^{x}} f(x, y)\,\mathrm{d}y\mathrm{d}x$$

Region $\quad R = \{(x,y) \in \mathbb{R}^2 ; \ x \in [0,2] \ , \ e^{-x} \leq y \leq e^{x}\}$



Aim : change the description of R so that we can integrate wrt x first, then y

Range of y in the region R : $\quad e^{-2} \leq y \leq e^{2}$

If $y \in [e^{-2}, 1]$ and $(x,y) \in R$, then

x is s.t $\quad e^{-x} \leq y \overset{\ln}{\iff} -x \leq \ln(y)$

$\iff x \geq -\ln(y)$

and we also have $x \leq 2$

$\Big\} \ -\ln(y) \leq x \leq 2$

If $y \in [1, e^{2}]$, then the condition on x becomes

$y \leq e^{x} \iff x \geq \ln y \quad (\text{and } x \leq 2)$

Integral

$$\int_R f(x,y)\,dx\,dy$$

$$= \boxed{\int_{e^{-2}}^{1} \left(\int_{-\ln(y)}^{2} f(x,y)\,dx \right) dy + \int_{1}^{e^2} \left(\int_{\ln(y)}^{2} f(x,y)\,dx \right) dy}$$

Switching order of integration (2)

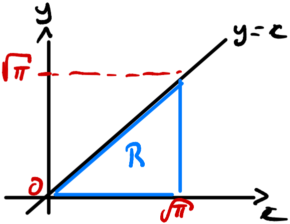
Changing the definition of R: We have

$$\begin{aligned} R &= \left\{(x,y) \in \mathbb{R}^2;\ x \in [0,2],\ e^{-x} \le y \le e^{x}\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2;\ y \in [e^{-2},1],\ -\ln(y) \le x \le 2\right\} \\ &\quad \bigcup \left\{(x,y) \in \mathbb{R}^2;\ y \in [1,e^{2}],\ \ln(y) \le x \le 2\right\} \end{aligned}$$

New formula for the integral:

$$\int\int_R f(x,y)\,\mathrm{d}A = \int_{e^{-2}}^{1}\int_{-\ln(y)}^{2} f(x,y)\,\mathrm{d}x\mathrm{d}y + \int_{1}^{e^2}\int_{\ln(y)}^{2} f(x,y)\,\mathrm{d}x\mathrm{d}y$$

Choosing order of integration (1)



Function:

$$z = f(x,y) = \sin\left(x^2\right)$$

Region: Of the form

$$R = \left\{(x,y) \in \mathbb{R}^2;\ y \in [0,\sqrt{\pi}],\ \ y \le x \le \sqrt{\pi}\right\}$$

Problem: Compute

$$\int\int_R f(x,y)\,\mathrm{d}A$$

Choosing order of integration (2)

Impossible computation: Write

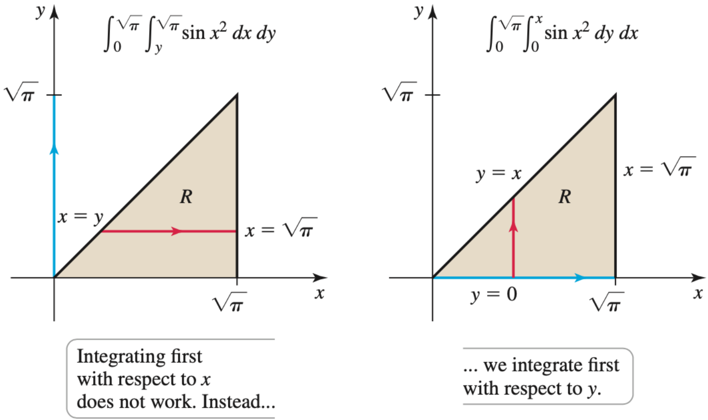
$$\int\int_R f(x,y)\,\mathrm{d}A = \int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}} \sin\left(x^2\right)\,\mathrm{d}x\mathrm{d}y$$

Then antiderivative of $\sin(x^2)$ not known!

Solution: Switch order of integration, ie write

$$R = \left\{(x,y)\in\mathbb{R}^2;\ x\in[0,\sqrt{\pi}],\ 0\le y\le x\right\}$$

Choosing order of integration (3)



Choosing order of integration (3)

Computing the integral:

$$\begin{aligned}
\int\int_R f(x,y)\,\mathrm{d}A &= \int_0^{\sqrt{\pi}}\int_0^{x} \sin\left(x^2\right)\,\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\sqrt{\pi}} \overbrace{(2)x\sin\left(x^2\right)}^{u'\sin u}\,\mathrm{d}x \qquad \text{with } u = x^2 \\
&= -\frac{1}{2}\cos\left(x^2\right)\Big|_0^{\sqrt{\pi}} \\
&= 1
\end{aligned}$$

Remark: This trick does not always work!