Outline

Summary 1st lecture

Two operations on vectors For $\vec{u}, \vec{v} \in \mathbb{R}^3$

$\underbrace{\vec{u} \cdot \vec{v}}_{\in \mathbb{R}}$ $\underbrace{\vec{u} \times \vec{v}}_{\in \mathbb{R}^3}$

Both are related to orthogonality

Dot product

$\vec{u} = \langle 1, 2, 3 \rangle$ $\vec{v} = \langle 4, 5, 6 \rangle$

Then $\vec{u} \cdot \vec{v} = 1 \times 4 + 2 \times 5 + 3 \times 6 = 32$

Important fact: $\vec{u} \perp \vec{v}$ iff $\vec{u} \cdot \vec{v} = 0$

Cross product $\vec{u} = \langle 2,1,1\rangle$ $\vec{v} = \langle 5,0,1\rangle$

Then

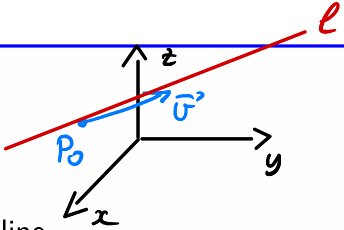
$$\vec{u}\times\vec{v} = \left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 2 & 1 & 1 & 2 & 1 \\ 5 & 0 & 1 & 5 & 0 \end{array}\right.$$

$$= \vec{i}\,(1 - 0) + \vec{j}\,(5 - 2) + \vec{k}\,(0 - 5) = \langle 1, 3, -5\rangle = \vec{n}$$

Important fact $\vec{n} \perp \vec{u}$ and $\vec{n} \perp \vec{v}$

Parametric form of the equation of a line

Proposition 8.



Let

- $P_0 = (x_0, y_0, z_0)$ point in $\mathbb{R}^3$
- $\mathbf{v} = \langle a, b, c \rangle$ vector

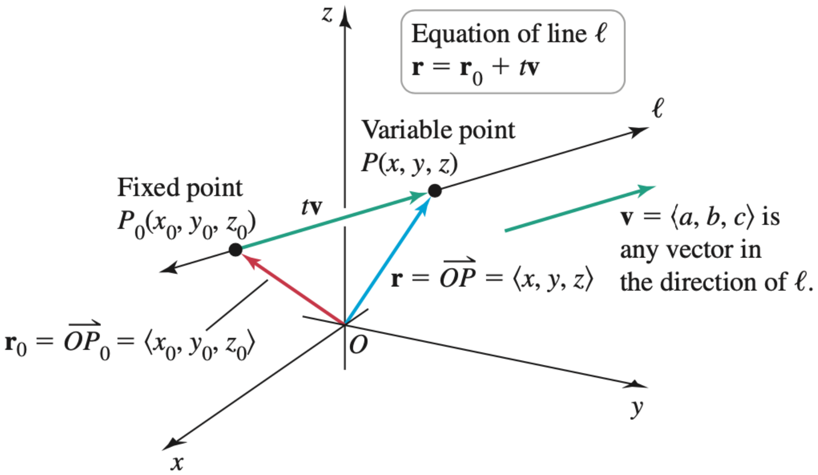
Then the parametric equation of a line passing through P_0 in the direction of $\mathbf{v}$ is

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle, \qquad t \in \mathbb{R}.$$

For coordinates, we get

$$\begin{cases} x &= x_0 + a\,t \\ y &= y_0 + b\,t \\ z &= z_0 + c\,t \end{cases}$$

Line in space: illustration



Example of parametric form (1)

Problem: Find the equation of a line

- Through point $(1, 2, 3)$
- Along $\mathbf{v} = \langle 4, 5, 6 \rangle$

Example

$P_0(1,2,3)$ $\vec{u} = \langle 4,5,6 \rangle$

Then the line going through P_0, with direction $\vec{u}$ has an eq. of the form

$$\langle x, y, z \rangle = \langle 1, 2, 3 \rangle + t \langle 4, 5, 6 \rangle$$

Coordinates

$$\begin{cases} x = 1 + 4t \\ y = 2 + 5t \\ z = 3 + 6t \end{cases}$$

Rmk Today we will see 2 descriptions of lines

(i) $P_0 + \vec{v}$

(ii) Intersection of 2 planes

In order to get from (ii) to (i), we will use $\cdot$ and $\times$

Example of parametric form (2)

Vector form:

$$\langle x, y, z\rangle = \langle 1, 2, 3\rangle + t\,\langle 4, 5, 6\rangle\,, \qquad t \in \mathbb{R}.$$

Coordinates form:

$$\begin{cases} x &= 1 + 4\,t \\ y &= 2 + 5\,t \\ z &= 3 + 6\,t \end{cases}$$

Example of line segment (1)

Problem: Find the equation of line segment

$$\text{From } P(0,1,2) \text{ to } Q(-3,4,7)$$

Example

Line going through $P(0,1,2)$ and $Q(-3,4,7)$

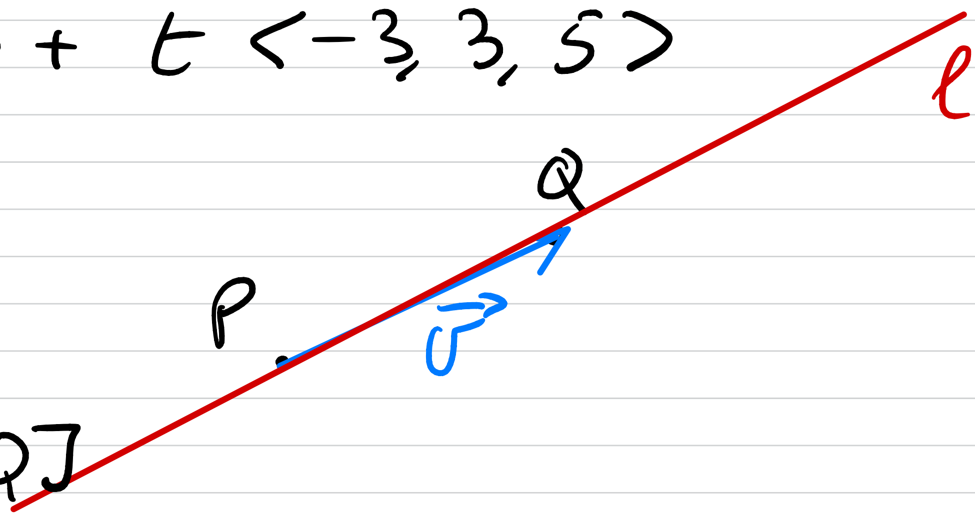
The direction for this line is

$$\vec{v} = \vec{PQ} = \langle -3-0,\ 4-1,\ 7-2 \rangle$$

$$= \langle -3,\ 3,\ 5 \rangle$$

Equation

$$\langle x, y, z \rangle = \langle 0, 1, 2 \rangle + t \langle -3, 3, 5 \rangle$$



Rmk

$t \in \mathbb{R}$: whole line

$t \in [0,1]$: segment $[P,Q]$

Example of line segment (2)

Direction vector: $\mathbf{v} = \vec{PQ} = \langle -3, 3, 5 \rangle$

Initial vector: $\vec{OP} = \langle 0, 1, 2 \rangle$

Equation:

$$\langle x, y, z \rangle = \langle 0, 1, 2 \rangle + t \langle -3, 3, 5 \rangle, \qquad t \in [0, 1].$$

Points of intersection for lines

Problem: $t \in \mathbb{R}$, $s \in \mathbb{R}$

Determine if ℓ_1 and ℓ_2 intersect and find point of intersection, with

$$\begin{aligned} \ell_1 &: \quad x = 2 + 3t, \quad y = 3t, \quad z = 1 - t \\ \ell_2 &: \quad x = 4 + 2s, \quad y = -3 + 3s, \quad z = -2s \end{aligned}$$

In order to know if there is an intersection, 2 steps:

(i) See if $\vec{v_1} \parallel \vec{v_2}$. If $\vec{v_1} \parallel \vec{v_2}$, no intersection

(ii) If $\vec{v_1} \nparallel \vec{v_2}$, try to solve the system

Points of intersection for lines (2)

Step 1: Check that $\mathbf{v}_1$ not parallel to $\mathbf{v}_2$. Here

$$\mathbf{v}_1 = \langle 3, 3, -1 \rangle, \quad \text{not parallel to} \quad \mathbf{v}_2 = \langle 2, 3, -2 \rangle$$

If $\vec{v}_1 \parallel \vec{v}_2$, then we would have $\vec{v}_1 = c\,\vec{v}_2$. Since the 2nd coord of v_1 and v_2 are the same, this constant would be $c = 1$. Thus we would have $\vec{v}_1 = \vec{v}_2$, which is not true.

Step 2: Equation for intersection

system with 2 unknown 3 equations ←

$$\begin{cases} 2 + 3t &= 4 + 2s \\ 3t &= -3 + 3s \\ 1 - t &= -2s \end{cases}$$

This system has no solution

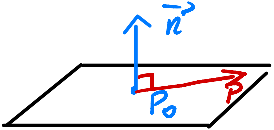
$\hookrightarrow$ ℓ_1 does not intersect ℓ_2

Points of intersection for lines (3)

Some conclusions:

1. If $\mathbf{v}_1 \parallel \mathbf{v}_2$,
   $\hookrightarrow$ ℓ_1 does not intersect ℓ_2
2. Even if $\mathbf{v}_1$ not parallel to $\mathbf{v}_2$,
   $\hookrightarrow$ we can have that ℓ_1 does not intersect ℓ_2
3. In the latter case, we say that the lines ℓ_1 and ℓ_2 are skewed

Equation of a plane in $\mathbb{R}^3$



$\overrightarrow{P_0P} \perp \vec{n}$. This can be written

$$\overrightarrow{P_0P} \cdot \vec{n} = 0$$

Proposition 9.

Let

- $P_0 = (x_0, y_0, z_0)$ point in $\mathbb{R}^3$
- $\mathbf{n} = \langle a, b, c \rangle$ vector

Then the parametric equation of a plane passing through P_0 with normal vector $\mathbf{n}$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Remarks on plane equations

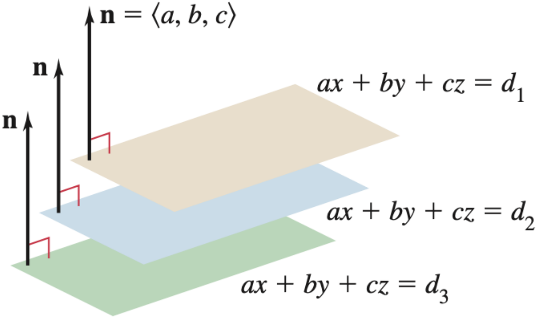
Plane and dot product: The plane is the set of points P such that

$$\vec{P_0 P} \cdot \mathbf{n} = 0$$

Other expression for the plane equation:

$$a\,x + b\,y + c\,z = d, \quad \text{with} \quad d = a\,x_0 + b\,y_0 + c\,z_0$$

Plane: illustration



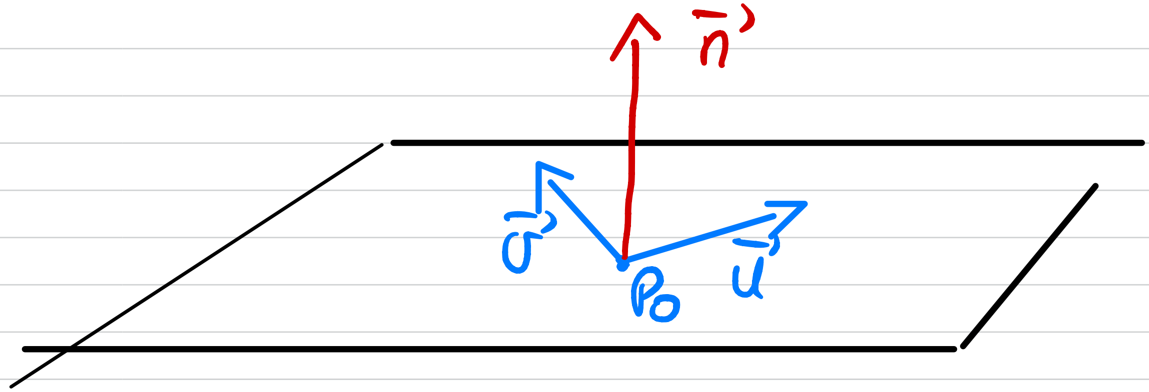
The normal vectors of parallel planes have the same direction.

Computing plane equations (1)

Problem: Compute the equation of the plane containing

$$\mathbf{u} = \langle 0, 1, 2 \rangle, \quad \mathbf{v} = \langle -1, 3, 0 \rangle, \quad P_0(-4, 7, 5)$$

$P_0(-4, 7, 5)$



Normal vector

$$\vec{n} = \vec{u} \times \vec{v} = -\langle 6, 2, -1 \rangle = \langle -6, -2, 1 \rangle$$

Equation : $6(x+4) + 2(y-7) - (z-5) = 0$

Computing plane equations (2)

Computing the normal vector:

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = -\langle 6, 2, -1 \rangle$$

Equation for the plane:

$$6x + 2y - z = -15$$