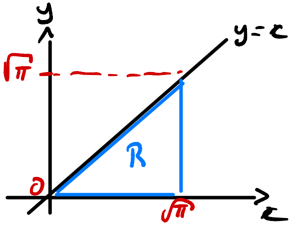
Choosing order of integration (1)



Function:

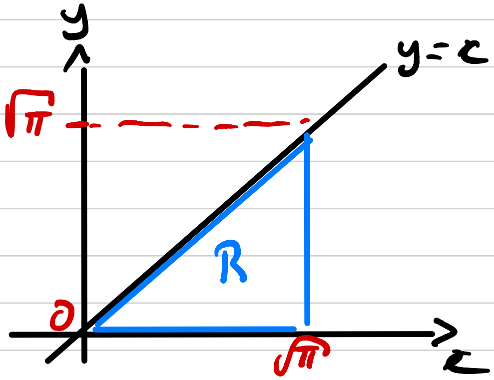
$$z = f(x, y) = \sin\left(x^2\right)$$

Region: Of the form

$$R = \left\{(x, y) \in \mathbb{R}^2;\ y \in [0, \sqrt{\pi}],\ y \leq x \leq \sqrt{\pi}\right\}$$

Problem: Compute

$$\int\int_R f(x, y)\, \mathrm{d}A$$



$$f(x,y) = \sin(x^2)$$

$$R = \{ 0 \le y \le \sqrt{\pi} \ , \ y \le x \le \sqrt{\pi} \}$$

Using the recipe,

$$\int_R f(x,y)\ dA = \int_0^{\sqrt{\pi}} \left(\int_y^{\pi} \sin(x^2)\ dx \right) dy$$

Problem : no expression for $\int \sin(x^2)\, dx$

Possible solution: switch x & y. For this, write

$R = \{ 0 \le x \le \sqrt{\pi},\ 0 \le y \le x \}$. Then

constant in y

$$\int_R f(x,y)\, dA = \int_0^{\sqrt{\pi}} \left(\int_0^x \sin(x^2)\, dy \right) dx$$

$$= \int_0^{\sqrt{\pi}} \sin(x^2) \times y \Big|_0^x \ dx = \int_0^{\sqrt{\pi}} x \sin(x^2)\, dx$$

Computation. We have obtained $u' \sin(u)$

$$\iint_R f(x,y)\, dA = \frac{1}{2}\int_0^{\sqrt{\pi}} (2x)\, \sin(x^2)\, dx$$

$$= \frac{1}{2} \quad -\cos(x^2)\Big|_0^{\sqrt{\pi}}$$

$$= \frac{1}{2}\left(-\cos(\pi) + \cos(0) \right)$$

$$= \frac{1}{2}\left(1 + 1 \right)$$

$$= 1$$

Choosing order of integration (2)

Impossible computation: Write

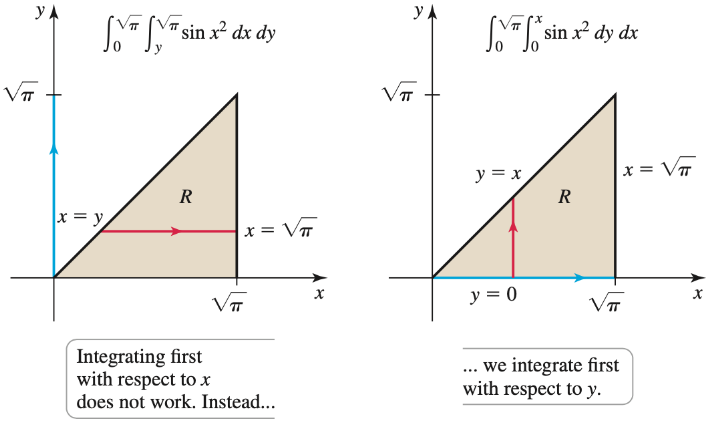
$$\int\int_R f(x,y)\,\mathrm{d}A = \int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}} \sin\left(x^2\right)\,\mathrm{d}x\mathrm{d}y$$

Then antiderivative of $\sin(x^2)$ not known!

Solution: Switch order of integration, ie write

$$R = \left\{(x,y)\in\mathbb{R}^2;\, x\in[0,\sqrt{\pi}],\; 0\le y\le x\right\}$$

Choosing order of integration (3)



Choosing order of integration (3)

Computing the integral:

$$\int\int_R f(x,y)\,\mathrm{d}A = \int_0^{\sqrt{\pi}} \int_0^x \sin\left(x^2\right)\,\mathrm{d}y\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{\sqrt{\pi}} \overbrace{(2)x\sin\left(x^2\right)}^{u' \sin u}\,\mathrm{d}x \quad \text{with } u = x^2$$

$$= -\frac{1}{2}\cos\left(x^2\right)\Big|_0^{\sqrt{\pi}}$$

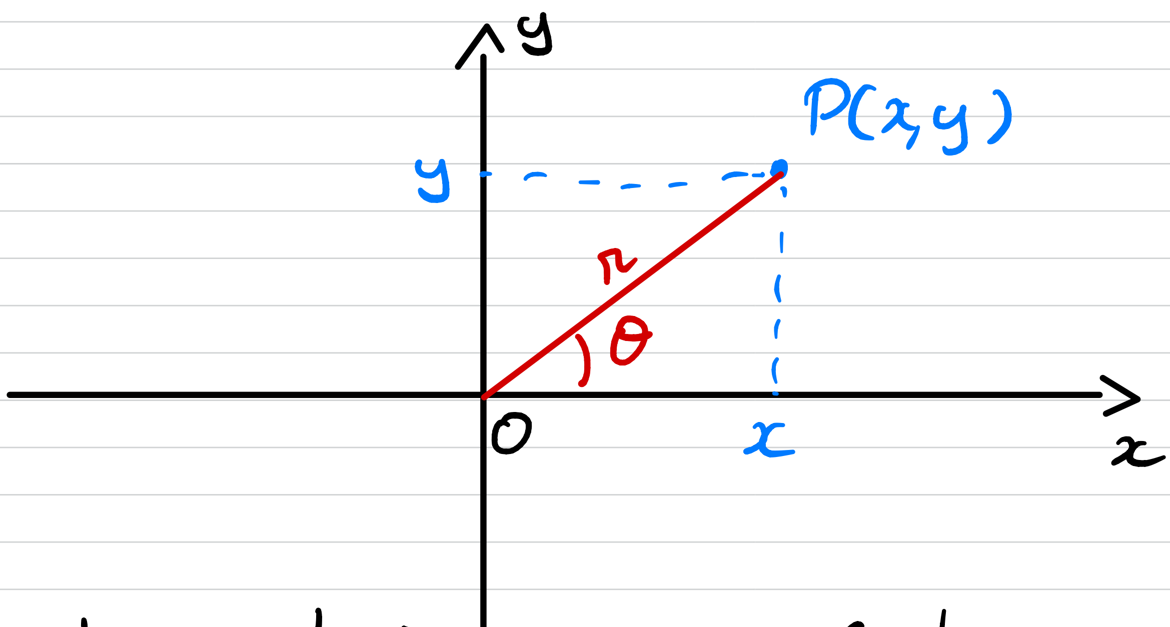
$$= 1$$

Remark: This trick does not always work!

Outline

Polar coordinates In $\mathbb{R}^2$, we have 2 main sets of coordinates

(i) Cartesian (ii) Polar



Polar to cartesian

$x = r\cos(\theta)$

$y = r\sin(\theta)$

Cartesian to polar

$r = (x^2 + y^2)^{\frac{1}{2}}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Recalling polar coordinates

Cartesian coordinates: (x, y)

Polar coordinates: (r, θ) with

- $r \equiv$ distance from origin
- $\theta \equiv$ angle wrt x-axis

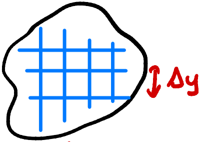
Polar to Cartesian: We have

$$x = r\cos(\theta), \qquad y = r\sin(\theta).$$

Cartesian to polar: We have

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Area of a small pizza crust (1)

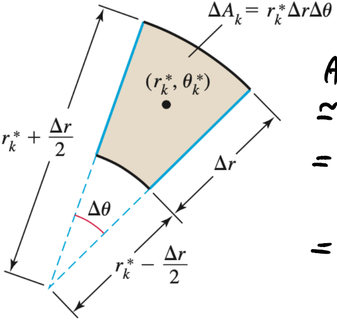


Recall: For integration in Cartesian coordinates

$\hookrightarrow$ We used area of small rectangles $\Delta x \Delta y$

Integral of $f \simeq \sum f(x_k, y_k)\, \Delta x\, \Delta y \longrightarrow \int_R f(x,y)\, dx dy$

New aim: Find area of a small rectangle in polar coordinates



Area of crust

Area crust

$\simeq$ Area of rectangle

$= \Delta r \times r \Delta\theta$ (arc length)

$= r\, \Delta r\, \Delta\theta$

Area of a small pizza crust (2)

Approximation: If Δr and $\Delta\theta$ are small, then

$$\begin{aligned} \text{Area(Pizza crust)} &\simeq \text{Area(Small rectangle)} \\ &= \Delta r \, (r \, \Delta\theta) \\ &= r \, \Delta r \, \Delta\theta \end{aligned}$$

Polar change of coordinates

Theorem 2.

Let

- $f(x, y)$ continuous function
- R polar region of the form

$$R = \{(r, \theta);\ a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$$

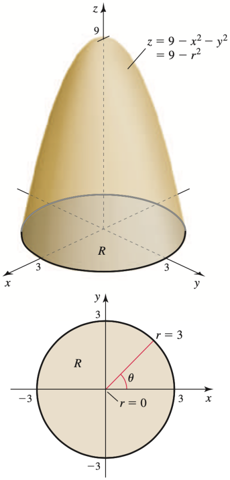
Then we have

$$\int\int_R f(x, y)\, \mathrm{d}A = \int_\alpha^\beta \int_a^b f\left(r\cos(\theta), r\sin(\theta)\right) r\, \mathrm{d}r \mathrm{d}\theta$$

Computing a volume (1)

Problem: Find the volume bounded by

- Paraboloid $z = 9 - x^2 - y^2$
- xy-plane



Function $f(x,y) = 9 - x^2 - y^2 \longrightarrow$ surface (paraboloid)

Intersection with xy-plane : when $z = 0$, i.e

$$9 - x^2 - y^2 = 0 \iff x^2 + y^2 = 9 \rightarrow \text{circle radius } 3$$

Thus the volume V is expressed as

$$V = \iint_R f(x,y)\, dA ,$$

where $R = \{ (x,y) \in \mathbb{R}^2 ; \ x^2 + y^2 \leq 3 \}$

Rmk Both f & R are better expressed in polar coordinates :

$$f(x,y) = 9 - x^2 - y^2 = 9 - r^2$$

$$R = \{ (r,\theta) ; \ 0 \leq r \leq 3 , \ 0 \leq \theta \leq 2\pi \}$$

In polar coordinates

$$f(x,y) = 9 - x^2 - y^2 = 9 - r^2$$

$$R = \{ (r,\theta) ; \; 0 \le r \le 3 \; , \; 0 \le \theta \le 2\pi \}$$

and

does not depend on θ

$$V = \int_0^{2\pi} \int_0^3 (9 - r^2)\, r \, dr \; d\theta$$

$$= 2\pi \int_0^3 (9 - r^2)\, r \, dr$$

$$= 2\pi \int_0^3 (9r - r^3)\, dr$$

$$= 2\pi \; \frac{9}{2} r^2 - \frac{r^4}{4} \Big|_0^3$$

$$\boxed{V = \frac{81}{2} \pi}$$

General fact

$$\int_a^b \int_c^d f(u)\, g(v)\, du\, dv$$

$$= \int_a^b f(u)\, du \times \int_c^d g(v)\, dv$$

Computing a volume (2)

Intersection with xy-plane: Circle defined by

$$x^2 + y^2 = 9$$

Polar coordinates domain:

$$R = \{(r, \theta);\ 0 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi\}$$

Computing a volume (3)

Volume as an integral: We have

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^3 \left(9 - r^2\right)\, \mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{9}{2}r^2 - \frac{1}{4}r^4\Big|_0^3 \,\mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{81}{4}\,\mathrm{d}\theta \end{aligned}$$

Thus

$$V = \frac{81\pi}{2}$$

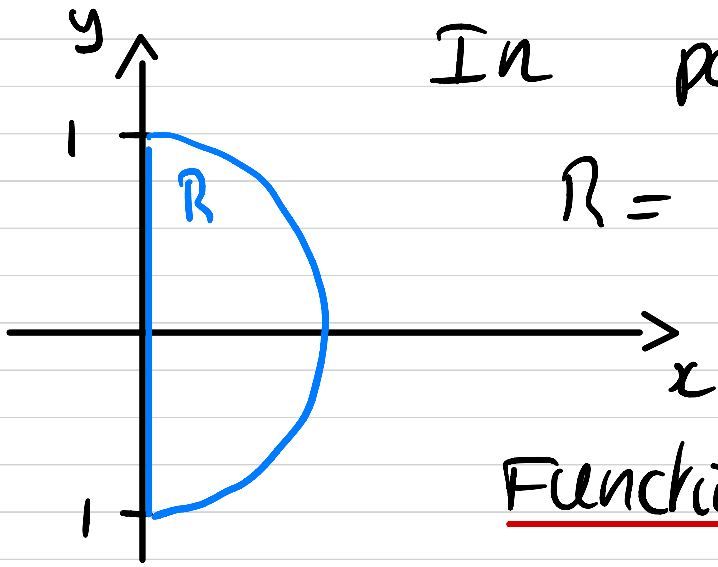
Example of polar integral (1)

Problem: Compute

$$I = \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \left(x^2 + y^2\right)^{3/2} \mathrm{d}x\mathrm{d}y$$

Remark:
The integral looks terrible in Cartesian coordinates!

Region $R = \{ -1 \le y \le 1 , \quad 0 \le x \le \sqrt{1-y^2} \}$



In polar coordinates

$$R = \{ 0 \le r \le 1 , \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \}$$

Function $f(x,y) = (x^2+y^2)^{3/2}$

$$= r^3$$

thus

depends on r only

$$\iint_R f(x,y)\, dA = \int_{-\pi/2}^{\pi/2} \int_0^1 r^3 \; r \; dr \; d\theta$$

$$= \pi \times \int_0^1 r^4 \, dr = \frac{\pi}{5}$$

Example of polar integral (2)

Domain in Cartesian coordinates:

$$R = \left\{ (x, y);\ -1 \le y \le 1,\ 0 \le x \le \sqrt{1 - y^2} \right\}$$

Domain in polar coordinates:

$$R = \left\{ (r, \theta);\ 0 \le r \le 1,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \right\}$$

Example of polar integral (3)

Integral in polar coordinates: We get

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r^3 \, r \, \mathrm{d}r \mathrm{d}\theta = \frac{\pi}{5}$$