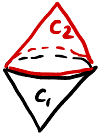
Volume between two cones (1)

Question: Will we always integrate the function 1 in $\mathbb{R}^3$?
Answer: NO. We just integrate 1 in order to compute volumes

Problem:

Compute the volume

- Above cone $C_1 : z = \sqrt{x^2 + y^2}$
- Below cone $C_2 : z = 2 - \sqrt{x^2 + y^2}$



Functions $C_1: \; z = (x^2+y^2)^{\frac{1}{2}} \quad = r$

$C_2: \; z = 2-(x^2+y^2)^{\frac{1}{2}} = 2-r$

Intersection when $r = 2-r \Leftrightarrow 2r = 2 \Leftrightarrow r=1$

Thus region of integration in the xy plane is $r \leq 1 \Leftrightarrow x^2+y^2 \leq 1$ (disk)

Integral

$$V = \iint_{\{x^2+y^2\leq 1\}} \int_{(x^2+y^2)^{\frac{1}{2}}}^{2-(x^2+y^2)^{\frac{1}{2}}} dz \; dx \; dy$$

$$= \iint_{\{x^2+y^2\leq 1\}} \left(2-2(x^2+y^2)^{\frac{1}{2}}\right) dx \; dy$$

Still ugly looking $\longrightarrow$ polar coordinates

$$V = \iint_{\{x^2+y^2\le 1\}} \left(2-2(x^2+y^2)^{\frac{1}{2}}\right)\,dx\,dy$$

In polar coordinates

Region of integration: $\{(r,\theta)\ ;\ 0\le\theta\le 2\pi,\ 0\le r\le 1\}$

Integral

$$V = 2\int_0^{2\pi}\int_0^1 (1-r)\,r\,dr\,d\theta$$

$$= 2\times 2\pi \int_0^1 (r-r^2)\,dr$$

$$= 4\pi \left.\frac{r^2}{2}-\frac{r^3}{3}\right|_0^1$$

$$= \frac{4}{6}\pi$$

$$\boxed{V = \frac{2\pi}{3}}$$

Volume between two cones (2)

Intersection of the 2 cones: Its projection on xy-plane is

$$x^2 + y^2 = 1$$

Strategy of integration:

- In xy-plane, surface delimited by $x^2 + y^2 = 1$
  $\hookrightarrow$ Easy domain (circle)

Conclusion: an easy way to integrate is in this order,

$$\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

Volume between two cones (3)

Integral computation: We get

$$\begin{aligned} V &= \int_{x^2+y^2\leq 1} \int_{\sqrt{x^2+y^2}}^{2-\sqrt{x^2+y^2}} \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_{x^2+y^2\leq 1} \left(2 - 2\sqrt{x^2+y^2}\right) \mathrm{d}y \, \mathrm{d}x \end{aligned}$$

Remark:
Terrible integral in Cartesian coordinates!

Volume between two cones (4)

Polar domain:

$$0 \leq r \leq 1, \qquad 0 \leq \theta \leq 2\pi$$

Volume in polar coordinates:

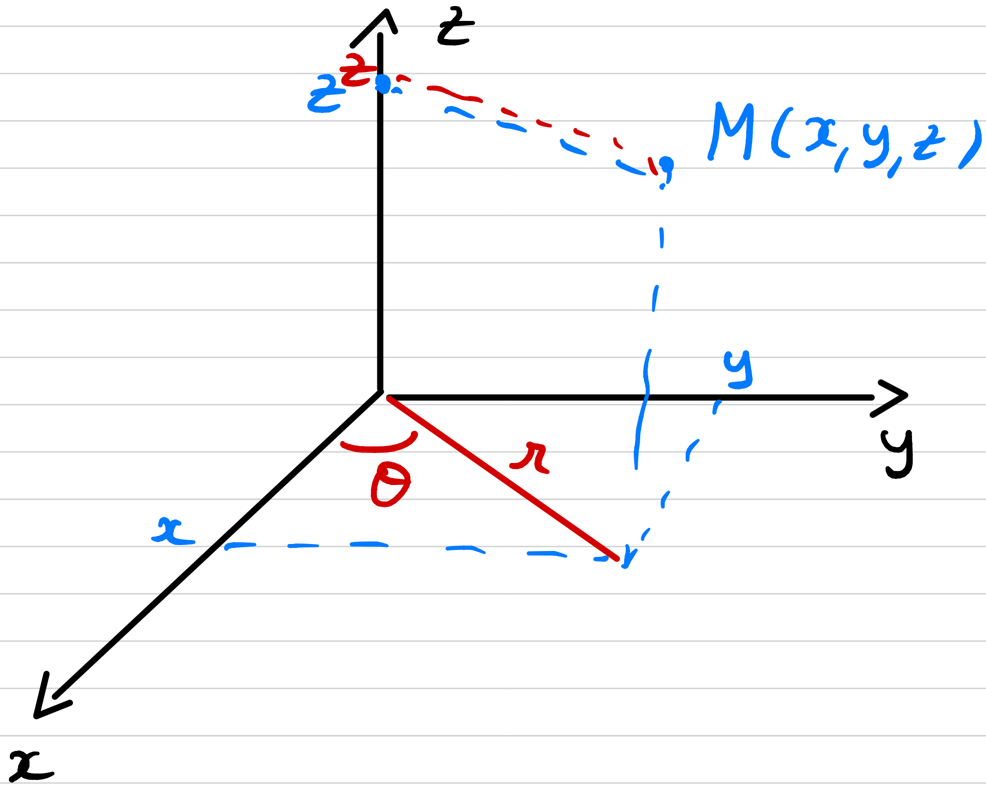
$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 (2 - 2r)\, r \,\mathrm{d}r\,\mathrm{d}\theta \\ &= 2\pi \times \frac{1}{3} \end{aligned}$$

We get

$$V = \frac{2\pi}{3}$$

Outline

Polar / Cartesian



Cartesian coord
(x, y, z)

Cylindrical coord
(r, θ, z)

Definition of cylindrical coordinates

Notation for cylindrical coordinates: Similar to polar coordinates

$$(r, \theta, z)$$

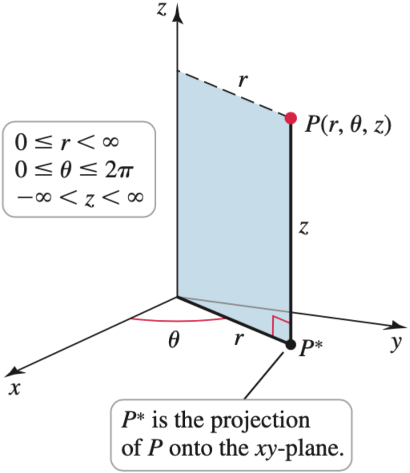
Conversion Cartesian to cylindrical:

$$r^2 = x^2 + y^2, \qquad \tan(\theta) = \frac{y}{x}, \qquad z = z$$

Conversion cylindrical to Cartesian:

$$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad z = z$$

Cylindrical coordinates: illustration



Example of cylindrical coordinates

Point in Cartesian coordinates:

$$P(-3, 3\sqrt{3}, 1)$$

Problem:

Find cylindrical coordinates for P

Answer:

$$\left(6, \frac{2\pi}{3}, 1\right)$$

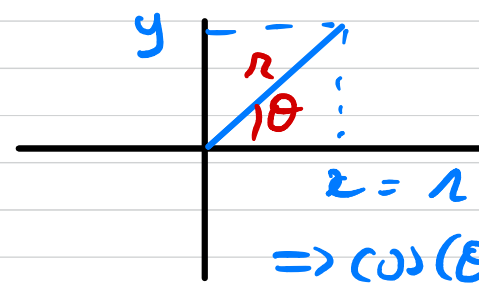
Point $P(\overbrace{-3}^{x}, \overbrace{3\sqrt{3}}^{y}, \overbrace{1}^{z})$ in **Cartesian**

Polar

$$r^2 = x^2 + y^2 = (-3)^2 + (3\sqrt{3})^2$$

$$= 9 + 27 = 3$$

$$\Rightarrow \quad r = \sqrt{36} = 6$$



$x = r\cos(\theta)$

$\Rightarrow \cos(\theta)$

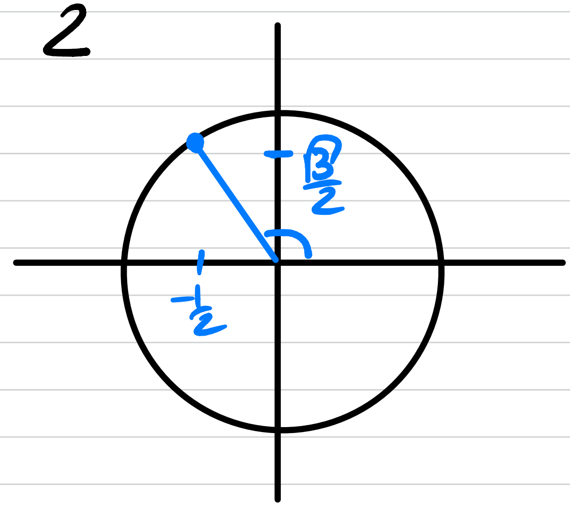
$= \dfrac{x}{r}$

Angle θ

$$\cos(\theta) = \frac{x}{r}$$

$$\cos(\theta) = \frac{-3}{6} = -\frac{1}{2}$$

$$\sin(\theta) = \frac{y}{r} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$$



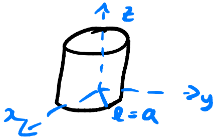
$$\Rightarrow \quad \theta = \frac{2\pi}{3}$$

Thus in polar, $\boxed{P\left(6, \frac{2\pi}{3}, 1\right)}$

Sets easily written in cylindrical coordinates

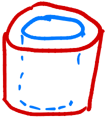
Cylinder:

$$r = a$$



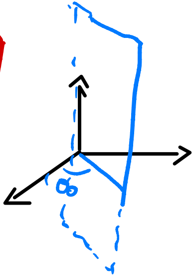
Cylindrical shell:

$$a \leq r \leq b$$



Vertical half plane:

$$\theta = \theta_0$$

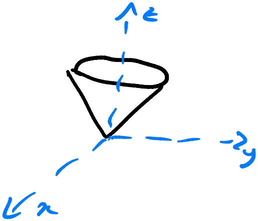


Horizontal plane:

$$z = a$$

Cone:

$$z = a\, r$$



Another domain in cylindrical coordinates (1)

Domain:

$$D = \left\{ (r, \theta, z);\ r^2 \leq z \leq 4 \right\}$$

Problem:

Identify this domain

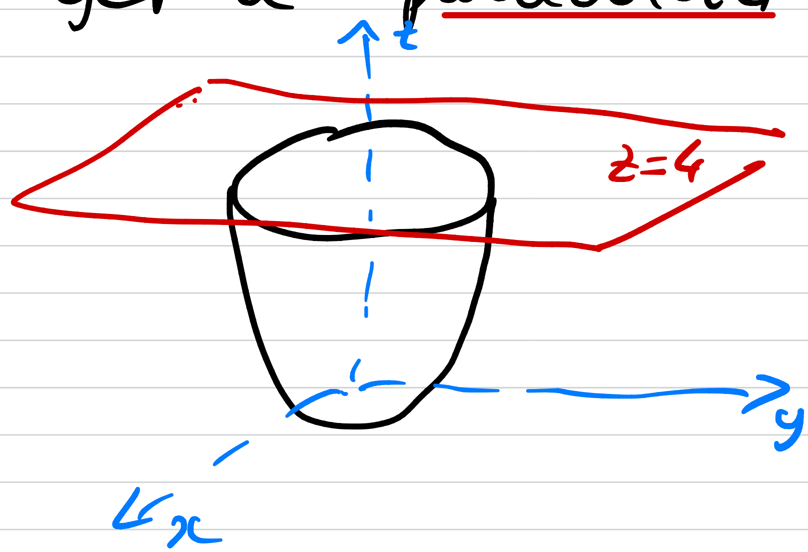
Domain $\{(r,\theta,z);\ r^2 \le z \le 4\}$ → simple to express

Surface $z = r^2 \iff z = x^2+y^2$

If $y = y_0$, we have $z = x^2 + y_0$
↳ parabola

If $z = z_0$, we have $x^2+y^2 = z_0$
↳ circle

we get a **paraboloid**



Another domain in cylindrical coordinates (2)

Lower bound on z: Given by the surface

$$z = r^2 \quad \Longleftrightarrow \quad z = x^2 + y^2$$

This is a paraboloid

Upper bound on z: Given by the surface

$$z = 4$$

This is a horizontal plane

Integration in cylindrical coordinates

Basic formula: In cylindrical coordinates (r, θ, z),

$$\int\int\int_D f(x, y, z)\,\mathrm{d}V = \int\int\int_D f(r\cos(\theta), r\sin(\theta), z)\, r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$$

When to use cylindrical coordinates: If

1. The domain D is one of the cylinder type domains
   $\hookrightarrow$ mentioned before
2. f is a function of $x^2 + y^2$, or f is a function of z

Example of cylindrical integral (1)

Problem: Compute

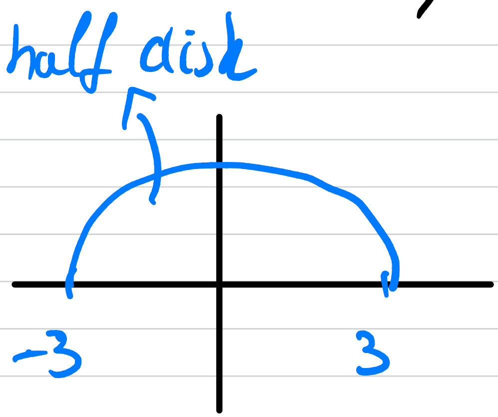
$$I = \int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\, \mathrm{d}z \mathrm{d}y \mathrm{d}x$$

Preliminary remark:
Awful integral in Cartesian coordinates!

Domain of integration

$$-3 \leq x \leq 3, \quad 0 \leq y \leq (9-x^2)^{\frac{1}{2}}$$

$$, \quad 0 \leq z \leq 9-x^2-y^2$$



In cylindrical coord

$$D = \left\{ \begin{array}{l} 0 \leq r \leq 3 \\ 0 \leq \theta \leq \pi \\ 0 \leq z \leq 9-r^2 \end{array} \right\}$$

Integral

$$I = \int_0^{\pi} \int_0^{3} \int_0^{9-r^2} r \times r \, dz \, dr \, d\theta$$

$$= \pi \int_0^{3} (9-r^2)\, r^2 \, dr = \cdots = \frac{162\pi}{5}$$

Example of cylindrical integral (2)

Domain:

$$-3 \leq x \leq 3 \quad \text{and} \quad 0 \leq y \leq \sqrt{9 - x^2}$$
$$\Longleftrightarrow$$
$$0 \leq r \leq 3 \quad \text{and} \quad 0 \leq \theta \leq \pi$$

Computing the integral: With cylindrical coordinates,

$$I = \int_0^{\pi} \int_0^{3} \int_0^{9-r^2} r \, \mathrm{d}z \, r \, \mathrm{d}r \mathrm{d}\theta$$

We get

$$I = \frac{162\pi}{5}$$