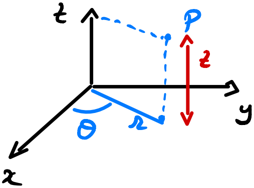
Mass of a solid paraboloid (1)



cyl coord: (r, θ, z)

Definition of the solid: Bounded by

- Paraboloid $z = 4 - r^2$
- Plane $z = 0$

Problem: Find mass of solid if density is

$$f(r, \theta, z) = 5 - z$$

(lighter as one goes up)

Intersection with xy-plane

$z=0 \iff 4-r^2=0$

$\iff r=2 \quad (r\geq 0)$

(disk with radius 2)

Domain

$$D=\{0\leq\theta\leq 2\pi,\ 0\leq r\leq 2,\ 0\leq z\leq 4-r^2\}$$

Mass

$$M=\int_0^{2\pi}\int_0^2\int_0^{4-r^2}(5-z)\ r\ dz\ dr\ d\theta$$

$$=2\pi\int_0^2\left(5z-\frac{z^2}{2}\Big|_0^{4-r^2}\right)r\ dr$$

polynomial

$$=2\pi\int_0^2\left\{5(4-r^2)-\frac{(4-r^2)^2}{2}\right\}r\ dr = \dots = \frac{88}{3}\pi$$

Mass of a solid paraboloid (2)

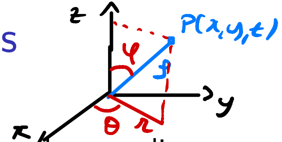
Domain: We have

$$D = \left\{0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 4 - r^2\right\}$$

Mass: We compute

$$\begin{aligned}
M &= \int\int\int_D f(r,\theta,z)\,\mathrm{d}V \\
&= \int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} (5-z)\,\mathrm{d}z\, r\,\mathrm{d}r\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{2} \left(24r - 2r^3 - r^5\right)\ r\,\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{44}{3}\,\mathrm{d}\theta = \frac{88\pi}{3}
\end{aligned}$$

Definition of spherical coordinates



Notation for spherical coordinates: Similar to polar coordinates

rho

$$(\rho, \varphi, \theta), \quad \text{with} \quad \rho \geq 0, \quad 0 \leq \varphi \leq \pi, \quad \underbrace{0 \leq \theta \leq 2\pi}_{\text{like cylindrical coord.}}$$

phi theta

Conversion Cartesian to spherical:

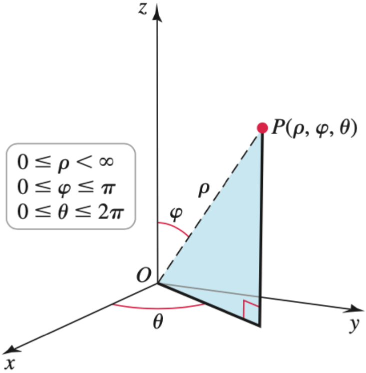
$$\rho^2 = x^2 + y^2 + z^2, \qquad + \text{ trigonometry to find } \varphi, \theta$$

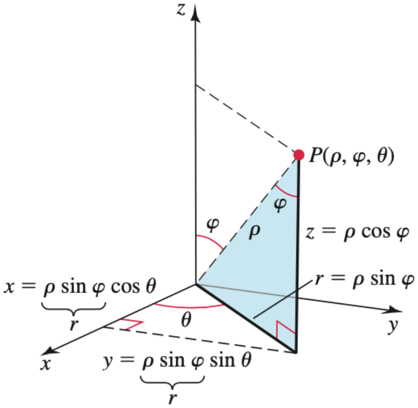
Conversion spherical to Cartesian:

$$x = \rho \sin(\varphi) \cos(\theta), \qquad y = \rho \sin(\varphi) \sin(\theta), \qquad z = \rho \cos(\varphi)$$

$$r = \rho \sin\varphi \Rightarrow x = r\cos(\theta) = \rho \sin(\varphi) \cos(\theta)$$
$$y = r \sin(\theta) = \rho \sin(\varphi) \sin(\theta)$$

Spherical coordinates: illustration





Example of spherical coordinates

Point in spherical coordinates:

$$P\left(1, \frac{\pi}{6}, \frac{\pi}{3}\right)$$

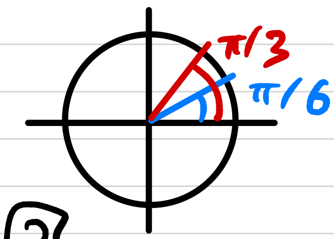
Problem:

Find Cartesian coordinates for P

Answer:

$$\left(\frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$$

Point in spherical $\quad P(\rho = 1,\ \varphi = \frac{\pi}{6},\ \theta = \frac{\pi}{3})$



In cartesian

$$z = \rho \cos(\varphi) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$x = \rho \sin(\varphi) \cos(\theta) = \sin\left(\frac{\pi}{6}\right) \cos\left(\frac{\pi}{3}\right)$$

$$x = \frac{1}{2} \times \frac{1}{4} \quad = \frac{1}{4}$$

$$y = \rho \sin(\varphi) \sin(\theta) = \sin\left(\frac{\pi}{6}\right) \sin\left(\frac{\pi}{3}\right)$$

$$y = \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$$

$$\boxed{P\left(\frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)}$$

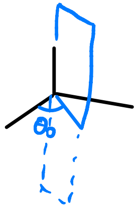
Sets easily written in spherical coordinates

Sphere:

$$\rho = a$$

Vertical half plane:

$$\theta = \theta_0$$



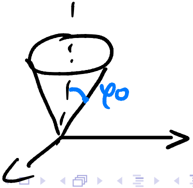
Horizontal plane:

$$\rho = a \sec(\varphi)$$

$$z = a \iff \rho \cos(\varphi) = a$$
$$\iff \rho = \frac{a}{\cos(\varphi)} = a \sec(\varphi)$$

Cone:

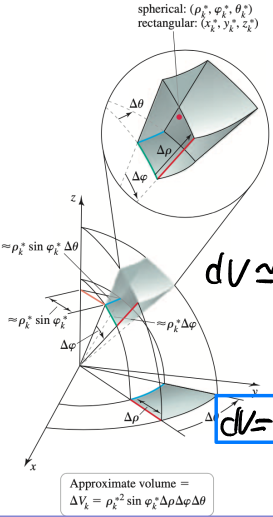
$$\varphi = \varphi_0$$

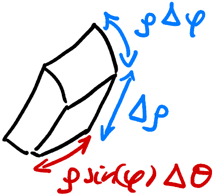


Small spherical volume

Formula: We have

$$\mathrm{d}V = \rho^2 \sin(\varphi)\, \mathrm{d}\rho\, \mathrm{d}\theta\, \mathrm{d}\varphi$$





Thus

$$dV \simeq \rho \sin(\varphi) \Delta\theta \times \Delta\rho \times \rho \Delta\varphi$$

$$\boxed{dV = \rho^2 \sin(\varphi)\, \Delta\rho\, \Delta\theta\, \Delta\varphi}$$

Integration in spherical coordinates

Basic formula: In spherical coordinates (r, θ, z),

$$\int\int\int_D f(x,y,z)\,\mathrm{d}V$$
$$= \int\int\int_D f\,(\underbrace{\rho\cos(\theta)\sin(\varphi)}_{x}, \underbrace{\rho\sin(\theta)\sin(\varphi)}_{y}, \underbrace{\rho\cos(\varphi)}_{z})\ \underbrace{\rho^2\sin(\varphi)\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi}_{dV}$$

When to use spherical coordinates: If

1. The domain D is one of the spherical type domains
   $\hookrightarrow$ mentioned before
2. f is a function of $x^2 + y^2 + z^2$

Example of spherical integral (1)

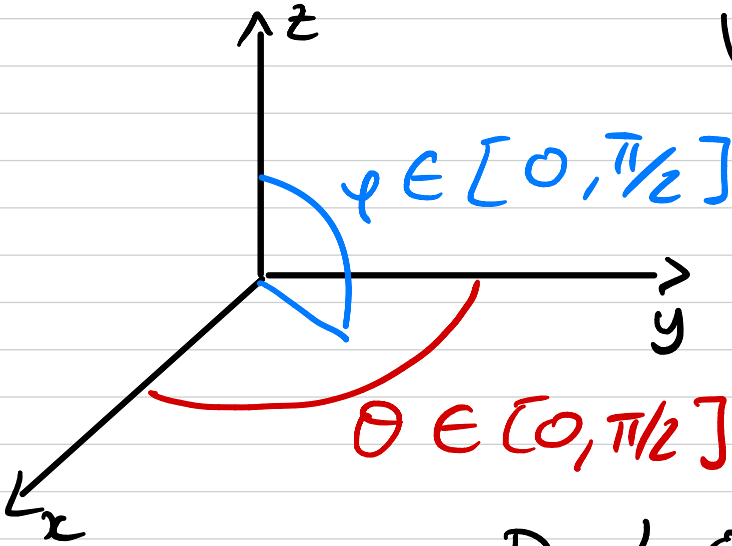
Domain: We consider

D = region in the first octant between two spheres of radius 1 and 2 centered at the origin.

Problem: Compute

$$I = \int\int\int_D \left(x^2 + y^2 + z^2\right)^{-3/2} \mathrm{d}V$$

$$\rho = (x^2+y^2+z^2)^{1/2} \qquad \rho^{-3} = (x^2+y^2+z^2)^{-3/2}$$



When $x \geq 0,\ y \geq 0,\ z \geq 0$ (first octant)

we have

$$\varphi \in [0, \pi/2], \quad \theta \in [0, \pi/2]$$

Domain

$$D = \{ 0 \leq \theta \leq \pi/2,\ 1 \leq \rho \leq 2,\ 0 \leq \varphi \leq \tfrac{\pi}{2} \}$$

Integral

$$I = \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^{-3} \times \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= \frac{\pi}{2} \left(\int_0^{\pi/2} \sin(\varphi)\, d\varphi \right) \left(\int_1^2 \rho^{-1}\, d\rho \right)$$

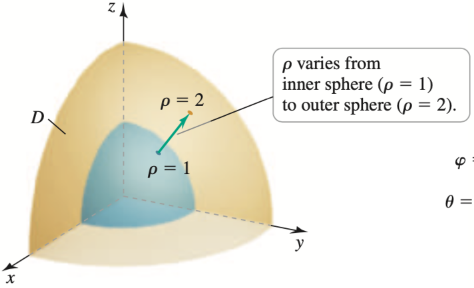
$$= \frac{\pi}{2} \underbrace{-\cos(\varphi)\Big|_0^{\pi/2}}_{1} \quad \underbrace{\ln(\rho)\Big|_1^2}_{\ln(2)}$$

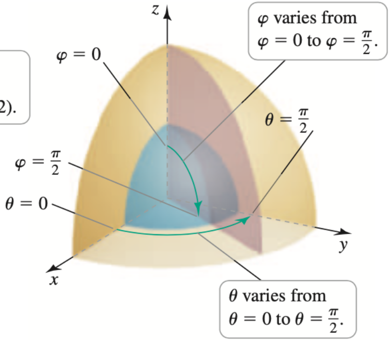
$$\boxed{I = \frac{\pi \ln(2)}{2}}$$

Example of spherical integral (2)

Expressing D in spherical coordinates:

$$D = \left\{ 1 \le \rho \le 2,\ 0 \le \varphi \le \frac{\pi}{2},\ 0 \le \theta \le \frac{\pi}{2} \right\}$$





Example of spherical integral (3)

Integral in spherical coordinates:

$$I = \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^{-3} \, \rho^2 \sin(\varphi) \, \mathrm{d}\rho \, \mathrm{d}\varphi \, \mathrm{d}\theta$$

Computation:

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^{\pi/2} \ln(\rho)\Big|_1^2 \sin(\varphi) \, \mathrm{d}\varphi \, \mathrm{d}\theta \\ &= \ln(2) \int_0^{\pi/2} \left(-\cos(\varphi)\right)\Big|_0^{\pi/2} \mathrm{d}\theta \\ &= \frac{\ln(2)\,\pi}{2} \end{aligned}$$