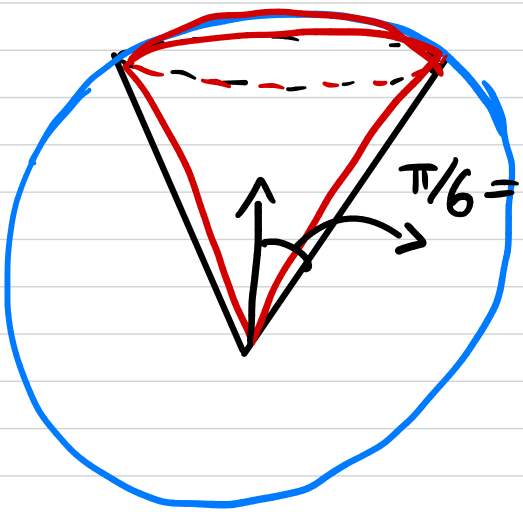
Volume of an ice cream cone (1)

Domain: We consider

$$D = \text{region between cone } \varphi = \frac{\pi}{6} \text{ and sphere } \rho = 4.$$

Problem: Compute

$$V = \text{Volume of } D = \int\int\int_D \mathrm{d}V$$



Domain

$$D = \{ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi/6,\ 0 \le \rho \le 4 \}$$

Volume

$$V = \int_0^{2\pi} \int_0^{\pi/6} \int_0^{4} 1 \times \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= 2\pi \int_0^{\pi/6} \sin(\varphi)\, d\varphi \int_0^4 \rho^2\, d\rho$$

$$= 2\pi \left. -\cos(\varphi) \right|_0^{\pi/6} \left. \frac{\rho^3}{3} \right|_0^4$$

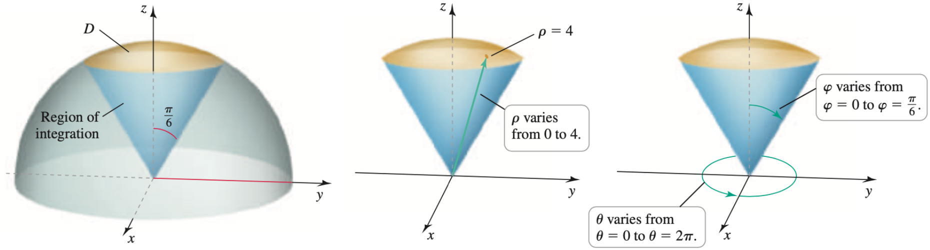
$$= 2\pi \left(1 - \frac{\sqrt{3}}{2} \right) \times \frac{64}{3}$$

$$\boxed{V = \frac{64\pi\,(2-\sqrt{3})}{3}}$$

Volume of an ice cream cone (2)

Expressing D in spherical coordinates:

$$D = \left\{0 \le \rho \le 4,\ 0 \le \varphi \le \frac{\pi}{6},\ 0 \le \theta \le 2\pi\right\}$$



Volume of an ice cream cone (3)

Integral in spherical coordinates:

$$I = \int_0^{2\pi} \int_0^{\pi/6} \int_0^4 \rho^2 \sin(\varphi)\, \mathrm{d}\rho\, \mathrm{d}\varphi\, \mathrm{d}\theta$$

Computation:

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^{\pi/6} \frac{\rho^3}{3}\Big|_0^4 \sin(\varphi)\, \mathrm{d}\varphi\, \mathrm{d}\theta \\ &= \frac{64}{3} \int_0^{2\pi} (-\cos(\varphi))\Big|_0^{\pi/6} \mathrm{d}\theta \\ &= \frac{64}{3}\left(1 - \frac{\sqrt{3}}{2}\right) 2\pi \\ &= \frac{64\pi(2-\sqrt{3})}{3} \end{aligned}$$

Outline

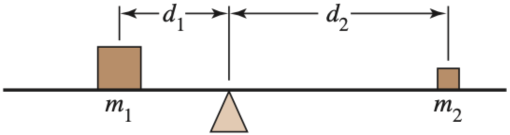
1. Double integrals over rectangular regions
2. Double integrals over general regions
3. Double integrals in polar coordinates
4. Triple integrals
5. Triple integrals in cylindrical and spherical coordinates
6. Integrals for mass calculations

A playground example (1)

Ci - Ga

Seesaw principle: Seesaw in equilibrium if

$$m_1 d_1 = m_2 d_2$$



Another look at equilibrium

$x_1 < \bar{x} < x_2$

Question: coordinate of $\bar{x}$ in terms of x_1, x_2 ?

Equating the forces we get

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$\Leftrightarrow (m_1 + m_2)\bar{x} = m_1 x_1 + m_2 x_2$$

$$\Leftrightarrow \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

center of mass

total moment

total mass

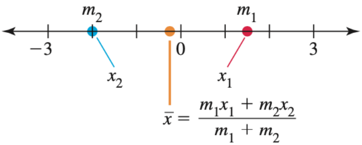
A playground example (2)

Notation:

Call $\bar{x}$ the center of mass for the 2-body seesaw system

Seesaw principle revisited: Seesaw in equilibrium if

$$m_1 (x_1 - \bar{x}) = m_2 (\bar{x} - x_2)$$



Solving for $\bar{x}$: We get

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{\text{Average(mass} \times \text{distance)}}{\text{Average(mass)}}$$

Center of mass of a ~~3~~ 2-d body

Theorem 4.

Let

- D closed bounded region in $\mathbb{R}^3$
- $\rho =$ Density function on D
- Mass of D given by $m = \int\int\int_D \rho(x, y, \cancel{z})\, \mathrm{d}V$ 2d-case

Then the coordinates of center of mass for D are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\int\int\int_D x\, \rho(x, y, \cancel{z})\, \mathrm{d}V$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\int\int\int_D y\, \rho(x, y, \cancel{z})\, \mathrm{d}V \quad \text{2d-case}$$

$$\cancel{\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\int\int\int_D z\, \rho(x, y, z)\, \mathrm{d}V}$$

Moments

distance to xy-plane

mass density

Definition of moment: In center of mass definition, the quantity

$$M_{xy} = \int\int\int_D z\,\rho(x,y,z)\,\mathrm{d}V$$

is called moment with respect to the xy-plane.

Remark: Moments are of the form

$$\text{Average}(\text{mass} \times \text{distance})$$

A 2-d example (1)

Domain: We consider $D \subset \mathbb{R}^2$ defined by

$$R = \left\{(x, y); 1 \leq x^2 + y^2 \leq 4\right\} \bigcap \text{ First quadrant}$$

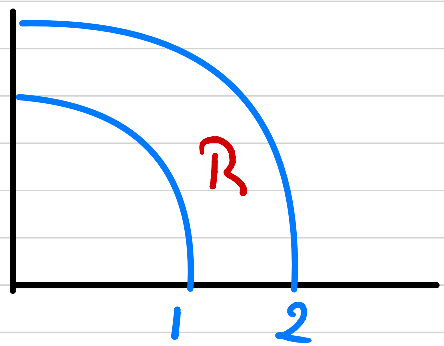
Density of mass: Given by

$$\rho(x, y) = \sqrt{x^2 + y^2}$$

Problem:

Find the center of mass of this object

Region $R = \{ 1 \le x^2+y^2 \le 4 \} \cap 1^{st}$ quadrant



In polar

$R = \{ 1 \le r \le 2 \; , \; 0 \le \theta \le \pi/2 \}$

Density $\rho = (x^2+y^2)^{1/2} = r$

Mass

$$m = \iint_R \rho \, dx\,dy \overset{\text{polar}}{=} \int_0^{\pi/2} \int_1^2 r \; r \, dr \, d\theta$$

$$m = \frac{\pi}{2} \int_1^2 r^2 \, dr = \frac{\pi}{2} \left. \frac{r^3}{3} \right|_1^2$$

$$m = \frac{7\pi}{6}$$

Center of mass

$\bar{y}$

$$\bar{x} = \frac{1}{m}\iint_R x\,\rho(x,y)\,dx\,dy$$

Polar

$\sin(\theta)$

$$= \frac{1}{m}\int_0^{\pi/2}\int_1^2 r\cos(\theta)\times r\times r\,dr\,d\theta$$

$$= \frac{1}{m}\left(\int_0^{\pi/2}\cos(\theta)\,d\theta\right)\int_1^2 r^3\,dr$$

$$= \frac{1}{m}\sin(\theta)\Big|_0^{\pi/2}\quad\frac{r^4}{4}\Big|_1^2$$

$$= \frac{6}{7\pi}\times 1\times\frac{(16-1)}{4}$$

check

$$\bar{x} = \frac{45}{14\pi}\simeq 1.023 = \bar{y}$$

A 2-d example (2)

Total mass: We get (with convenient polar coordinates)

$$\begin{aligned} m &= \int\int_R \rho \, \mathrm{d}A \\ &= \int_0^{\pi/2}\int_1^2 \rho \, r \, \mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{\pi/2}\int_1^2 r^2 \, \mathrm{d}r\mathrm{d}\theta \end{aligned}$$

Thus

$$m = \frac{7\pi}{6}$$

A 2-d example (3)

Center of mass on the y-axis: We have

$$
\begin{aligned}
\bar{y} &= \frac{1}{m} \int\int_R y\,\rho\,\mathrm{d}A \\
&= \frac{6\pi}{7} \int_0^{\pi/2} \int_1^2 r \sin(\theta) \rho\, r\,\mathrm{d}r\mathrm{d}\theta \\
&= \frac{6\pi}{7} \int_0^{\pi/2} \int_1^2 r^3 \sin(\theta)\,\mathrm{d}r\mathrm{d}\theta \\
&= \frac{45}{14\pi}
\end{aligned}
$$

Thus

$$\bar{y} \simeq 1.023$$

A 2-d example (4)

Center of mass on the x-axis: We have

$$
\begin{aligned}
\bar{x} &= \frac{1}{m} \int\int_R x\,\rho\,\mathrm{d}A \\
&= \frac{6\pi}{7} \int_0^{\pi/2} \int_1^2 r\cos(\theta)\rho\,r\,\mathrm{d}r\mathrm{d}\theta \\
&= \frac{6\pi}{7} \int_0^{\pi/2} \int_1^2 r^3\,\cos(\theta)\,\mathrm{d}r\mathrm{d}\theta \\
&= \frac{45}{14\pi}
\end{aligned}
$$

Thus

$$\bar{x} \simeq 1.023$$

A 2-d example (5)

Conclusion: The center of mass is

$$(\bar{x}, \bar{y}) = (\underline{1.023, 1.023})$$

Here the problem is symmetric in (x, y), thus $\bar{x} = \bar{y}$

A 3-d example (1)

Domain: We consider $D \subset \mathbb{R}^3$ bounded by

- Hemisphere with radius a
- xy-plane

distance to origin

Density of mass: Given by (object heavier close to the center)

density $\longleftarrow$ $$f(\rho, \varphi, \theta) = 2 - \frac{\rho}{a}$$

Problem:

Find the center of mass of this object

Domain Nicely defined in spherical

$$\{0 \le \rho \le a, \quad 0 \le \varphi \le \frac{\pi}{2}, \quad 0 \le \theta \le 2\pi\} = D$$

Mass

$$m = \iiint_D f(x, y, z)\, dx\, dy\, dz$$

Spher.

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \left(2 - \frac{\rho}{a}\right) \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

A 3-d example (2)

Graph of the situation: