Average temperature (1)

Situation:

- Circular plate

$$R = \left\{x^2 + y^2 = 1\right\}$$

- Temperature distribution in the plane:

$$T(x, y) = 100\left(x^2 + 2y^2\right)$$

Problem:

Compute the average temperature on the edge of the plate

Curve $\{ \overbrace{\langle \cos(t), \sin(t) \rangle}^{\vec{r}(t)}, \ 0 \le t \le 2\pi \} \equiv C$

Temperature $T(x,y) = 100(x^2 + 2y^2)$

Average temp.

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$

$|\vec{r}\,'(t)|^2 = \sin^2(t) + \cos^2(t) = 1$

$$\bar{T} = \underbrace{\frac{1}{\text{length}(C)}}_{= 2\pi} \int_C T(x,y)\, ds$$

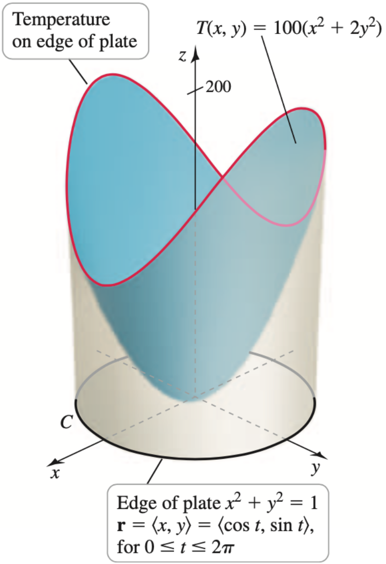
$$= \frac{1}{2\pi} \int_0^{2\pi} 100 \left(\cos^2(t) + 2\sin^2(t) \right) \overbrace{|\vec{r}\,'(t)|}^{=1} dt$$

$$= \frac{50}{\pi} \int_0^{2\pi} (1 + \sin^2(t))\, dt$$

$$= \frac{50}{\pi} \int_0^{2\pi} 1 + \left(\frac{1 - \cos(2t)}{2} \right) dt$$

$$= \frac{50}{\pi} \int_0^{2\pi} \left(\frac{3}{2} - \frac{1}{2} \overbrace{\cos(2t)}^{\text{intg is } 0} \right) dt = 150 = \bar{T}$$

Average temperature (2)



Average temperature (3)

Parametric description of C: $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$

Arc length: $|\mathbf{r}'(t)| = 1$

Line integral:

$$\begin{aligned} \int_C T(x,y)\,\mathrm{d}s &= 100 \int_0^{2\pi} \left(x(t)^2 + 2y(t)^2 \right) |\mathbf{r}'(t)|\,\mathrm{d}t \\ &= 100 \int_0^{2\pi} \left(\cos^2(t) + 2\sin^2(t) \right) \mathrm{d}t \\ &= 100 \int_0^{2\pi} \left(1 + \sin^2(t) \right) \mathrm{d}t \end{aligned}$$

Thus

$$\int_C T(x,y)\,\mathrm{d}s = 300\pi$$

Average temperature (4)

Recall:

$$\int_C T(x,y)\,\mathrm{d}s = 300\pi$$

Average temperature: Given by

$$\overline{T} = \frac{\int_C T(x,y)\,\mathrm{d}s}{\text{Length}(C)}$$

We get

$$\overline{T} = \frac{300\pi}{2\pi} = 150$$

Computation of line integrals in $\mathbb{R}^3$

Theorem 3.

We consider

- Curve C defined by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$
- Time interval $[a, b]$
- Arc length s of $\mathbf{r}$
- Function f defined on $\mathbb{R}^3$

Then we have

$$\int_C f \,\mathrm{d}s = \int_a^b f\left(x(t), y(t), z(t)\right) |\mathbf{r}'(t)| \mathrm{d}t$$

Example of line integral in $\mathbb{R}^3$ (1)

Situation:

- Two points in $\mathbb{R}^3$

$$P(1,0,0), \qquad Q(0,1,1)$$

- Function:

$$f(x,y,z) = xy + 2z$$

Problem: Compute $\int_C f(x,y)\,\mathrm{d}s$ in the following cases:

1. C is the segment from P to Q
2. C is the segment from Q to P

Curve: Segment from $P(1,0,0)$ to $Q(0,1,1)$

We have $\overrightarrow{PQ} = \langle -1, 1, 1 \rangle$

Segment: $\overbrace{\langle 1, 0, 0 \rangle}^{P} + t\, \overbrace{\langle -1, 1, 1 \rangle}^{\overrightarrow{PQ}}$, $t \in [0,1]$

$= \langle 1-t, t, t \rangle = \vec{r}(t)$, $t \in [0,1]$

Thus $\vec{r}\,'(t) = \langle -1, 1, 1 \rangle$

$|\vec{r}\,'(t)| = \sqrt{3}$

Line integral $\qquad f(x,y) = xy + 2z$

$$I = \int_C f(x,y)\, ds = \int_0^1 [(1-t)t + 2t]\sqrt{3}\, dt$$

$$= \sqrt{3}\int_0^1 (3t - t^2)\, dt = \sqrt{3}\left(\frac{3}{2} - \frac{1}{3}\right)$$

$$\boxed{I = \frac{7\sqrt{3}}{6}}$$

Curve : segment from $Q(0,1,1)$ to $P(1,0,0)$

we get $\vec{r}'(t) = \langle t, 1-t, 1-t \rangle$

$$|\vec{r}'(t)| = \sqrt{3}$$

we find (check)

$$I' = \int_0^1 f(x(t), y(t), z(t)) \, |\vec{r}'(t)| \, dt$$

$$I' = \frac{7\sqrt{3}}{6}$$

Conclusion The value of

$$\int_C f(x,y,z) \, ds \quad , \quad f(x,y,z) \in \mathbb{R}$$

does not depend on the parametrization of C. This is always true if f real-valued

Example of line integral in $\mathbb{R}^3$ (2)

Parametric equation for segment from P to Q:

$$\mathbf{r}(t) = \langle 1 - t, t, t \rangle \,, \qquad t \in [0, 1]$$

Arc length:

$$|\mathbf{r}'(t)| = \sqrt{3}$$

Example of line integral in $\mathbb{R}^3$ (3)

Line integral:

$$\begin{aligned}
\int_C f(x,y)\,\mathrm{d}s &= \int_C (xy + 2z)\ \mathrm{d}s \\
&= \int_0^1 ((1-t)t + 2t)\sqrt{3}\,\mathrm{d}t \\
&= \sqrt{3}\int_0^1 \left(3t - t^2\right)\ \mathrm{d}t \\
&= \sqrt{3}\left(\frac{3}{2} - \frac{1}{3}\right)
\end{aligned}$$

Thus we get

$$\int_C f(x,y)\,\mathrm{d}s = \frac{7\sqrt{3}}{6}$$

Example of line integral in $\mathbb{R}^3$ (4)

Parametric equation for segment from Q to P:

$$\mathbf{r}(t) = \langle t, 1-t, 1-t \rangle$$

Arc length: $|\mathbf{r}'(t)| = \sqrt{3}$

Line integral: One can check that we also have

$$\int_C f(x,y)\,\mathrm{d}s = \frac{7\sqrt{3}}{6}$$

General conclusion:

The value of $\int_C f(x,y)\,\mathrm{d}s$
does not depend on the parametrization of C

Line integral of a vector field

Definition 4.

We consider

- Curve $C : \mathbf{r}(s) = \langle x(s), y(s), z(s) \rangle$
- C is parametrized by arc length s
- $\mathbf{T}(s)$ unit tangent vector
- Vector field $\mathbf{F}$ defined on $\mathbb{R}^3$ → $\vec{F}(x,y,z) \in \mathbb{R}^3$

Then the line integral of $\mathbf{F}$ over C is

$$\int_C \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s$$ ↗ Not always easy to compute

Motivation: Line integrals crucial to compute work of a force $\mathbf{F}$

Computing line integrals

Theorem 5.

We consider

- Curve $C : \mathbf{r}(t) = \langle x(t), y(t), \cancel{z(t)} \rangle$
- C is parametrized by $t \in [a, b]$
- Vector field $\mathbf{F}$ defined on $\mathbb{R}^{\cancel{3}\,2}$

Then the line integral of $\mathbf{F}$ over C is given by

$$\int_C \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s = \int_C \mathbf{F}(t) \cdot \mathbf{r}'(t) \, \mathrm{d}t$$

Example of line integral for a vector field (1)

Situation:

- Two points in $\mathbb{R}^2$:

$$P(0,1), \qquad Q(1,0)$$

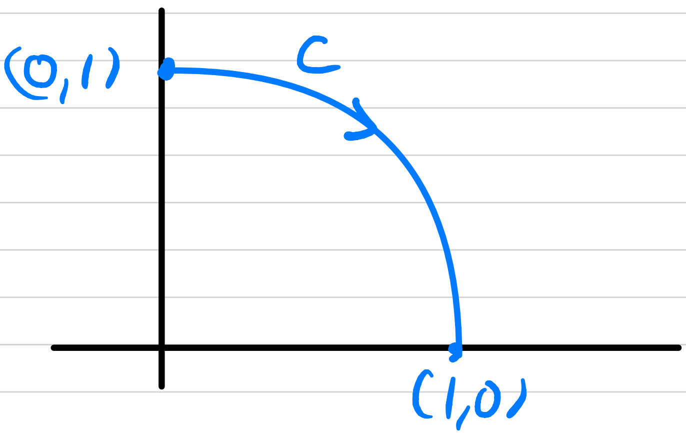
- Vector field:

$$\mathbf{F}(x,y) = \langle y - x,\, x \rangle$$

Problem: Compute $\int_C \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s$ in the following cases:

1. C_1 quarter-circle from P to Q
2. $-C_1$ quarter-circle from Q to P
3. C_2 path defined by segments $P(0,1)$–$O(0,0)$–$Q(1,0)$

$\vec{F}(x,y) = \langle y-x, x \rangle$



Curve

$C = \{ \overbrace{\langle \sin(t), \cos(t) \rangle}^{\vec{r}'(t)} ; 0 \le t \le \frac{\pi}{2} \}$

$\vec{r}''(t) = \langle \cos(t), -\sin(t) \rangle$

Line integral

$$I = \int_C \vec{F} \cdot \vec{T} \, ds$$

$$= \int_0^{\pi/2} \langle \cos(t) - \sin(t), \sin(t) \rangle \cdot \langle \cos(t), -\sin(t) \rangle \, dt$$

$$= \int_0^{\pi/2} [\cos^2(t) - \sin(t)\cos(t) - \sin^2(t)] \, dt$$

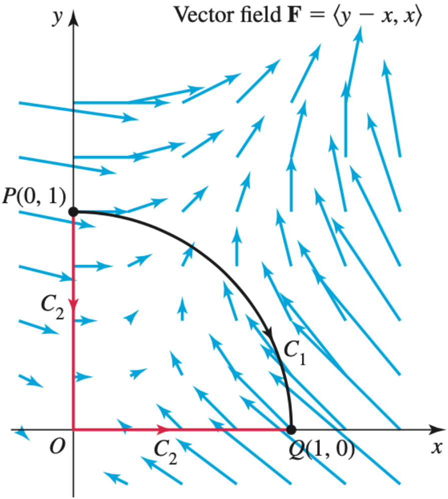
$$= \int_0^{\pi/2} [\cos^2(t) - \sin^2(t) - \sin(t)\cos(t)] \, dt$$

$$= \int_0^{\pi/2} [\cos(2t) - \frac{1}{2}\sin(2t)] \, dt$$

$$\boxed{I = -\frac{1}{2}}$$

Example of line integral for a vector field (2)

length (C)
$= \int_C 1 \, |\vec{r}'(t)| dt$



Example of line integral for a vector field (3)

Parametric equation for C_1:

$$\mathbf{r}(t) = \langle \sin(t), \cos(t) \rangle$$

Parametric equation for $\mathbf{F}$: Along C_1 we have

$$\mathbf{F} = \langle y - x,\ x \rangle = \langle \cos(t) - \sin(t), \sin(t) \rangle$$

Dot product: We have

$$\mathbf{F}(t) \cdot \mathbf{r}'(t) = \cos^2(t) - \sin^2(t) - \sin(t)\cos(t) = \cos(2t) - \frac{1}{2}\sin(2t)$$

Example of line integral for a vector field (4)

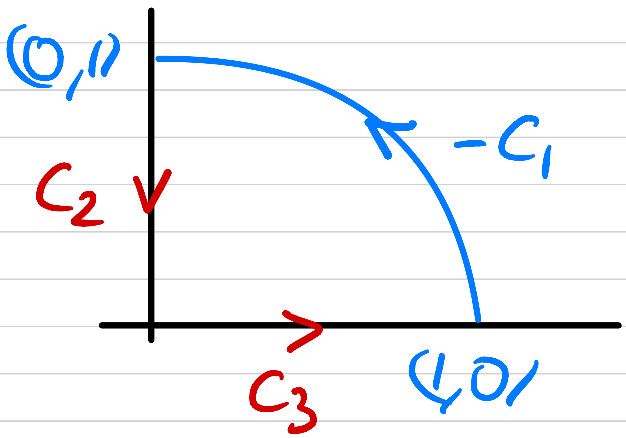
Line integral:

$$\begin{aligned}\int_{C_1} \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s &= \int_{C_1} \mathbf{F}(t)\cdot\mathbf{r}'(t)\,\mathrm{d}t \\ &= \int_0^{\pi/2} \left(\cos(2t) - \frac{1}{2}\sin(2t)\right)\,\mathrm{d}t \\ &= \frac{1}{2}\sin(2t) + \frac{1}{4}\cos(2t)\Big|_0^{\pi/2}\end{aligned}$$

Thus we get

$$\int_{C_1} \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s = -\frac{1}{2}$$

If we move along $-C_1$:



$$-C_1 : \{ \underbrace{(\cos(t), \sin(t))}_{\vec{r}'(t)} ;\ 0 \le t \le \tfrac{\pi}{2} \}$$

one finds

$$\int_{-C_1} \vec{F} \cdot \vec{T}\, ds = +\frac{1}{2}$$

Check at home

$$\int_{C_2} \vec{F} \cdot \vec{T}\, ds + \int_{C_3} \vec{F} \cdot \vec{T}\, ds = \int_{C_1} \vec{F} \cdot \vec{T}\, ds = -\frac{1}{2}$$

This is not always true, but true for a large class of vector field

Example of line integral for a vector field (5)

Line integral along $-C_1$: We find

$$\int_{-C_1} \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s = \frac{1}{2} = -\int_{C_1} \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s$$

Changing the orientation of C_1 changes the sign of the line integral

Line integral along C_2: We find

$$\int_{C_2} \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s = -\frac{1}{2} = \int_{C_1} \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s$$

Question: is this true for a large class of **F**?