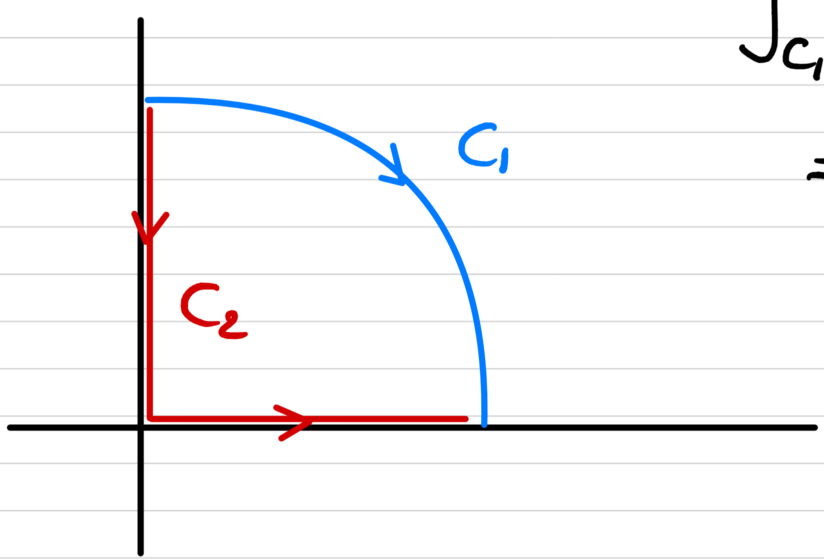
Outline

Last time

We have computed $\int_C \vec{F} \cdot \vec{r}\,' \, ds$

We have seen an example of $\vec{F}$ in $\mathbb{R}^2$

s.t.



$$\int_{C_1} \vec{F} \cdot \vec{r}\,' \, ds = \int_{C_2} \vec{F} \cdot \vec{r}\,' \, ds$$

Question: Is that true for a large class of vector fields?

Main issues in this section

Two important questions: $\vec{F} = \nabla\varphi$

1. When can we say that a $\underbrace{\text{vector field}}_{}$ is the gradient of a function?
2. What is special with this kind of vector fields?

↳ They are conservative

Conservative vector field

Definition 6.

Let

- D domain of $\mathbb{R}^2$
- $\mathbf{F}$ vector field defined on D

Then $\mathbf{F}$ is a conservative vector field if

There exists φ such that $\mathbf{F} = \nabla\varphi$ on D

Criterion for being conservative in $\mathbb{R}^2$

Notation: For $\varphi : \mathbb{R}^2 \to \mathbb{R}$, set $\varphi_x = \frac{\partial \varphi}{\partial x}$ and $\varphi_y = \frac{\partial \varphi}{\partial y}$

Theorem 7.

Consider a vector field in $R \subset \mathbb{R}^2$:

$$\mathbf{F} = \langle f, g \rangle$$

Then there exists φ such that:

$$\nabla \varphi \equiv \langle \varphi_x, \varphi_y \rangle = \mathbf{F} \quad \text{on } R,$$

if and only if $\mathbf{F}$ satisfies:

Due to the fact that

$f_y = \varphi_{xy}$

$g_x = \varphi_{yx}$

$$f_y = g_x \quad \text{on } R$$

Computation of function φ in $\mathbb{R}^2$

Aim: If $f_y = g_x$, find φ such that $\varphi_x = f$ and $\varphi_y = g$.

Recipe in order to get φ:

1. Write φ as antiderivative of f with respect to x:

   $\varphi(x,y) = \int f(x,y)\,dx + b(y)$ → the "constant" depends on y

   $$\varphi(x,y) = a(x,y) + b(y), \quad \text{where} \quad a(x,y) = \int f(x,y)\,dx$$

2. Get an equation for b by differentiating with respect to y:

   $$\varphi_y = g \iff b'(y) = g(x,y) - a_y(x,y)$$

3. Finally we get:

   $$\varphi(x,y) = a(x,y) + b(y).$$

<u>Vector field</u> $\vec{F} = \langle \overbrace{x+y}^{f}, \overbrace{x}^{g} \rangle$

<u>Is $\vec{F}$ conservative</u>? According to Thm 7, true if $f_y = g_x$. Here

$f_y = 1$, $g_x = 1$

$\Rightarrow$ $\vec{F}$ <u>conservative</u>, i.e there exists φ s.t. $\nabla\varphi = \vec{F}$.

Vector field $\vec{F} = \langle \overbrace{x+y}^{f}, \overbrace{x}^{g} \rangle$

In order to find φ s.t $\nabla\varphi = \vec{F}$,

① $\varphi(x,y) = \int (x+y)\, dx$ $\quad$ $\left(\varphi_x = f \Rightarrow \varphi = \int f\, dx \right)$

$\varphi(x,y) = \frac{1}{2} x^2 + xy + b(y)$

② We write $\varphi_y = g$

$\Leftrightarrow \cancel{x} + b'(y) = \cancel{x}$

$\Leftrightarrow b'(y) = 0 \Leftrightarrow b(y) = c$

③ $\boxed{\varphi(x,y) = \frac{1}{2} x^2 + xy + c}$

Example of conservative vector field (1)

Vector field:

$$\mathbf{F} = \langle x + y, x \rangle$$

Problem:

1. Is $\mathbf{F}$ conservative?
2. If $\mathbf{F}$ is conservative, compute φ such that $\nabla \varphi = \mathbf{F}$

Example of conservative vector field (2)

Recall:

$$\mathbf{F} = \langle x + y, x \rangle$$

Proof that **F** is conservative:

$$f_y = 1 = g_x$$

Thus **F** is conservative

Example of conservative vector field (3)

Computing φ: We have

$$\varphi = \int f(x, y)\,\mathrm{d}x + b(y) = \frac{1}{2}x^2 + yx + b(y)$$

Computing b: We write

$$\varphi_y = x \quad \Longleftrightarrow \quad x + b'(y) = x \quad \Longleftrightarrow \quad b'(y) = 0$$

Expression for φ: Since $b(y) = c$ for a constant c, we get

$$\varphi(x, y) = \frac{1}{2}x^2 + yx + c$$

Criterion for being conservative in $\mathbb{R}^3$

Theorem 8.

Consider a vector field in $R \subset \mathbb{R}^3$:

$$\mathbf{F} = \langle f, g, h \rangle$$

Then there exists φ such that:

$$\nabla \varphi \equiv \langle \varphi_x, \varphi_y, \varphi_z \rangle = \mathbf{F} \quad \text{on } R,$$

if and only if $\mathbf{F}$ satisfies:

$$f_y = g_x, \quad f_z = h_x, \quad g_z = h_y \quad \text{on } R$$

Computation of function φ in $\mathbb{R}^3$

$\varphi = \int f \, dx + b(y,z)$

Aim: If **F** is conservative, find φ
$\hookrightarrow$ such that $\varphi_x = f$, $\varphi_y = g$ and $\varphi_z = h$.

Recipe in order to get φ:

1. Write φ as antiderivative of f with respect to x:

$$\varphi(x, y) = a(x, y, z) + b(y, z), \quad \text{where} \quad a(x, y, z) = \int f(x, y, z) \, dx$$

2. Get an equation for b by differentiating with respect to y:

$$\varphi_y = g \quad \Longleftrightarrow \quad b_y(y, z) = g(x, y, z) - a_y(x, y, z)$$

3. Iterate this procedure with ∂_z

Example of conservative vector field in $\mathbb{R}^3$ (1)

Vector field:

$$\mathbf{F} = \left\langle x^2 - z\, e^y, y^3 - xz\, e^y, z^4 - x\, e^y \right\rangle$$

Problem:

1. Is $\mathbf{F}$ conservative?
2. If $\mathbf{F}$ is conservative, compute φ such that $\nabla\varphi = \mathbf{F}$

Vector field

$$\vec{F} = \langle \overbrace{x^2 - z e^y}^{f},\ \overbrace{y^3 - xz e^y}^{g},\ \overbrace{z^4 - x e^y}^{h} \rangle$$

Is $\vec{F}$ conservative? $\longrightarrow$ YES!

$$f_y = -z e^y \quad = \quad g_x = -z e^y$$

$$f_z = -e^y \quad = \quad h_x = -e^y$$

$$g_z = -x e^y \quad = \quad h_y = -x e^y$$

Vector field

$$\vec{F} = \langle \overbrace{x^2 - z e^y}^{f}, \overbrace{y^3 - xz e^y}^{g}, \overbrace{z^4 - x e^y}^{h} \rangle$$

Computing φ

① $\varphi(x,y,z) = \int f \, dx = \frac{1}{3} x^3 - xz e^y + b(y,z)$

② $\varphi_y = g$

$\Leftrightarrow \quad - xz e^y + b_y(y,z) = y^3 - xz e^y$

$\Leftrightarrow \quad b_y(y,z) = y^3$

Thus $\quad b(y,z) = \int y^3 \, dy = \frac{1}{4} y^4 + c(z)$

and

$$\varphi(x,y,z) = \frac{1}{3} x^3 - xz e^y + \frac{1}{4} y^4 + c(z)$$

$$\varphi(x,y,z) = \frac{1}{3}x^3 - xze^y + \frac{1}{4}y^4 + c(z)$$

③ We write $\varphi_z = h = z^4 - xe^y$

$$\varphi_z = h \iff -xe^y + c'(z) = z^4 - xe^y$$

$$\iff c'(z) = z^4$$

Thus $c(z) = \int z^4 \, dz = \frac{1}{5}z^5 + C$

We have obtained

$$\boxed{\varphi(x,y,z) = \frac{1}{3}x^3 - xze^y + \frac{1}{4}y^4 + \frac{1}{5}z^5 + C}$$

Check $\nabla\varphi = \vec{F}$

Example of conservative vector field in $\mathbb{R}^3$ (2)

Recall:

$$\mathbf{F} = \left\langle x^2 - z\,e^y, y^3 - xz\,e^y, z^4 - x\,e^y \right\rangle$$

Proof that **F** is conservative:

$$\begin{aligned} f_y = g_x &= -x\,e^y \\ f_z = h_x &= -e^y \\ g_z = h_y &= -x\,e^y \end{aligned}$$

Thus **F** is conservative

Example of conservative vector field in $\mathbb{R}^3$ (3)

Computing φ: We have

$$\varphi = \int f(x, y, z)\,\mathrm{d}x + b(y, z) = \frac{1}{3}x^3 - xz\,e^y + b(y, z)$$

Computing b: We write

$$\begin{aligned} \varphi_y = y^3 - xz\,e^y \quad &\Longleftrightarrow \quad -xz\,e^y + b_y = y^3 - xz\,e^y \\ &\Longleftrightarrow \quad b(y, z) = \frac{1}{4}y^4 + c(z) \end{aligned}$$

We have thus obtained

$$\varphi = \frac{1}{3}x^3 - xz\,e^y + \frac{1}{4}y^4 + c(z)$$

Example of conservative vector field in $\mathbb{R}^3$ (4)

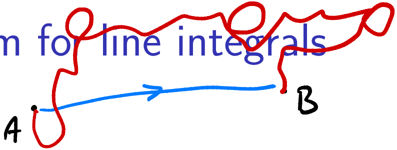
Computing c: We write

$$\varphi_z = z^4 - x\,e^y \iff -x\,e^y + c'(z) = z^4 - x\,e^y$$
$$\iff c(z) = \frac{1}{5}z^5 + d$$

Expression for φ: For a constant d, we get

$$\varphi(x, y, z) = \frac{1}{3}x^3 - xz\,e^y + \frac{1}{4}y^4 + \frac{1}{5}z^5 + d$$

Fundamental theorem for line integrals



Theorem 9.

Consider

- A conservative vector field $\mathbf{F}$ on $R \subset \mathbb{R}^3$
- φ such that $\nabla\varphi = \mathbf{F}$
- A piecewise smooth oriented curve $C \subset R$ from A to B

Then we have

$$\int_C \mathbf{F} \cdot \mathbf{T}\, \mathrm{d}s = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \varphi(B) - \varphi(A)$$