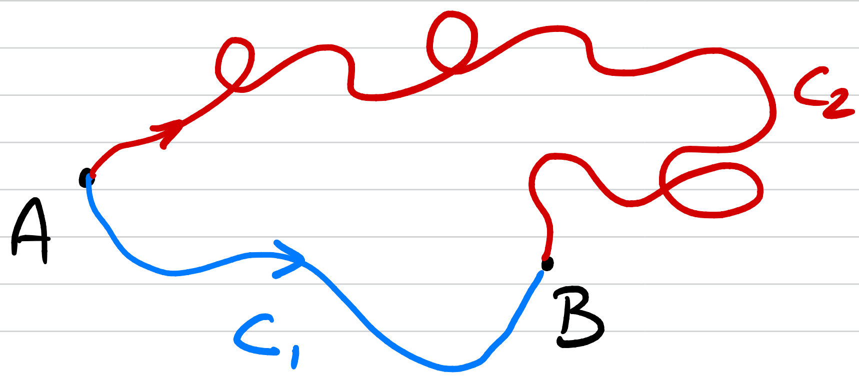
Thm If $\vec{F}$ s.t. $\vec{F} = \nabla\varphi$ (conservative v.f)

then

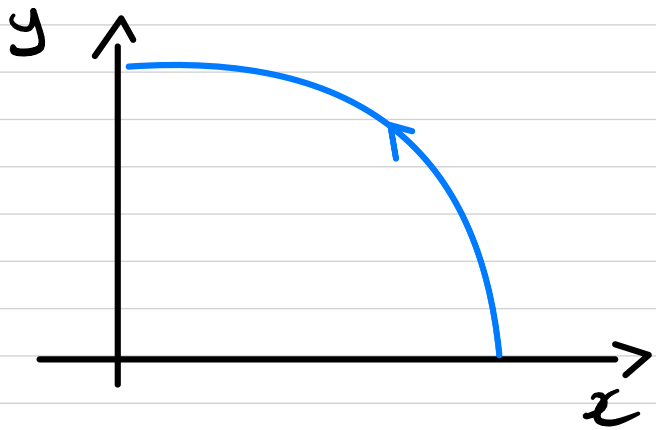
$$\int_C \vec{F} \cdot \vec{r}\,' \, dt = \varphi(B) - \varphi(A)$$

If C is any curve from A to B



Vector field

$$\vec{F} = \langle x, -y \rangle$$



$$C = \{ \underbrace{\langle \cos(t), \sin(t) \rangle}_{\vec{r}(t)} ; 0 \le t \le \tfrac{\pi}{2} \}$$

We have

$$\int_C \vec{F} \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{\pi/2} \langle \cos(t), -\sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \, dt$$

$$= \int_0^{\pi/2} -2 \sin(t) \cos(t) \, dt$$

$$= - \int_0^{\pi/2} \sin(2t) \, dt$$

$$= \frac{1}{2} \cos(2t) \Big|_0^{\pi/2}$$

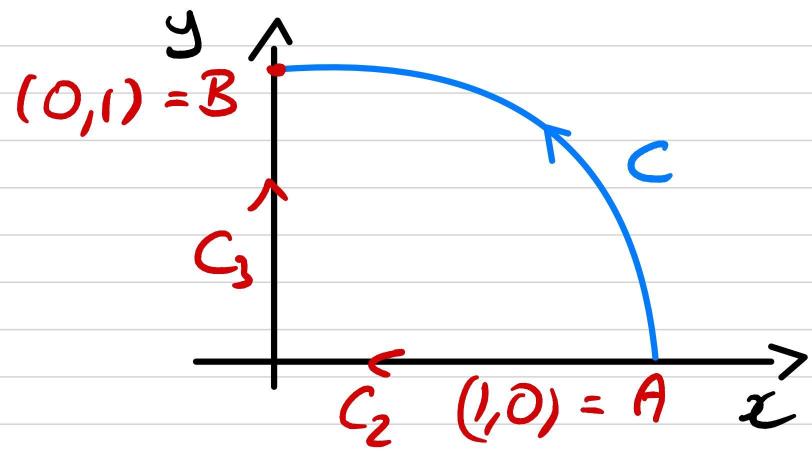
$$= -1$$

$\langle f, g \rangle = \vec{F} = \nabla\varphi$ iff $f_y = g_x$

Vector field $\vec{F} = \langle x, -y \rangle$

It can be shown (see recipe on Wednesday) that $\vec{F} = \nabla\varphi$ with

$$\varphi = \frac{1}{2}(x^2 - y^2)$$



For C we have

$\varphi(B) - \varphi(A)$

$= \frac{1}{2}(0^2 - 1^2) - \frac{1}{2}(1^2 - 0^2)$

$= -1$

We have found $\boxed{\int_C \vec{F} \cdot \vec{r}\,' \, dt = \varphi(B) - \varphi(A)}$

Verifying path independence (1)

Vector field:

$$\mathbf{F} = \langle x, -y \rangle$$

Curves: We consider

- C_1 quarter circle $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, \pi/2]$
- C_2 line $\mathbf{r}(t) = \langle 1 - t,\ t \rangle$ for $t \in [0, 1]$
- Both C_1 and C_2 go from $A(1,0)$ to $B(0,1)$

Problem:

1. Compute $\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ directly
2. Compute $\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$
   $\hookrightarrow$ using the fundamental theorem for line integrals

Verifying path independence (2)

Computation along C_1: We have

$$\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle, \qquad \mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle$$

Thus

$$\begin{aligned} \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^{\pi/2} \langle \cos(t), -\sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \, \mathrm{d}t \\ &= \int_0^{\pi/2} (-\sin(2t)) \, \mathrm{d}t \end{aligned}$$

We get

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -1$$

Verifying path independence (3)

Computation along C_2: We have

$$\mathbf{r}(t) = \langle 1-t,\, t\rangle\,, \qquad \mathbf{r}'(t) = \langle -1,\, 1\rangle$$

Thus

$$\int_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_0^1 \langle 1-t,\, t\rangle \cdot \langle -1,\, 1\rangle\, \mathrm{d}t = \int_0^1 (-1)\, \mathrm{d}t$$

We get

$$\int_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = -1$$

Verifying path independence (4)

Computing the potential φ: We have

$$\varphi(x,y) = \frac{1}{2}\left(x^2 - y^2\right) \quad \Longrightarrow \quad \nabla\varphi = \mathbf{F}$$

Using the fundamental theorem for line integrals: We have

$$\int_{C_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \varphi(0,1) - \varphi(1,0)$$

Thus we get

$$\int_{C_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = -1$$

Outline

2-dimensional curl

Definition 10.

Let

- $\mathbf{F} = \langle f, g \rangle$ vector field in $\mathbb{R}^2$

Then we define

$$\mathrm{Curl}(\mathbf{F}) = g_x - f_y$$

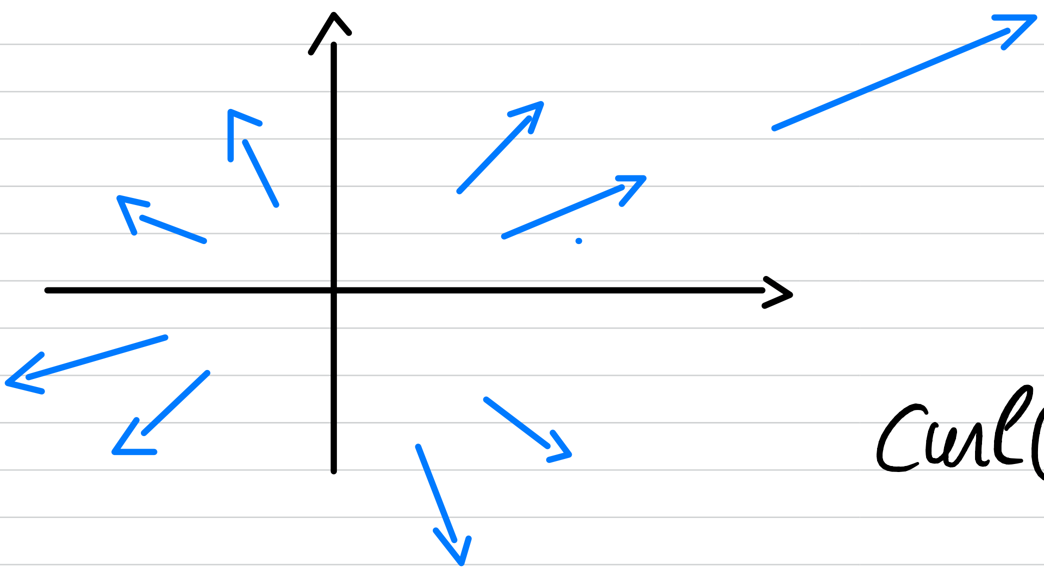
Another notation: In order to prepare the $\mathbb{R}^3$ version one can write

$$\mathrm{Curl}(\mathbf{F}) = (g_x - f_y)\,\mathbf{k}$$

Interpretation: $\mathrm{Curl}(\mathbf{F})$ represents

$\hookrightarrow$ The amount of rotation in $\mathbf{F}$

Vector field $\vec{F} = \langle \underbrace{x}_{f}, \underbrace{y}_{g} \rangle$



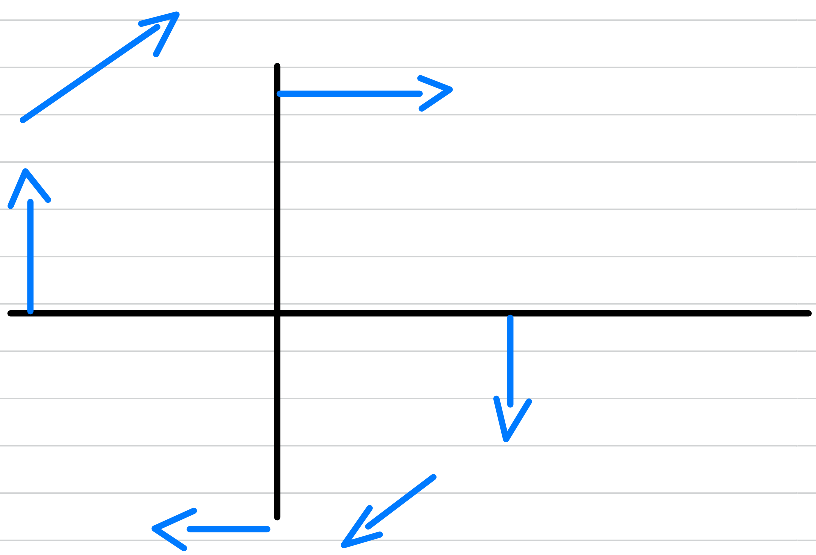
Looks like $\vec{F}$ doesn't have a lot of rotation

$$\text{Curl}(\vec{F}) = g_x - f_y$$
$$= 0 - 0$$
$$= 0$$

Here no rotation $\Rightarrow$ $\text{curl}(\vec{F}) = 0$

vector field

$$\vec{F} = \langle \overset{f}{y}, \overset{g}{-x} \rangle$$



It looks like $\vec{F}$ has some rotation

$$\operatorname{Curl}(\vec{F})$$
$$= g_x - f_y$$
$$= -1 - 1$$
$$= -2$$

Here we get rotation $\Rightarrow$ nontrivial Curl

Rmk $\operatorname{Curl}(\vec{F}) = g_x - f_y = 0$ if $\vec{F}$ is conservative

Example of irrotational vector field

Vector field: **F** defined by

$$\mathbf{F} = \langle x, \, y \rangle$$

Curl of **F**: We get

$$\mathrm{Curl}(\mathbf{F}) = g_x - f_y = 0$$

Interpretation: **F** has no rotational component
$\hookrightarrow$ **F** is said to be irrotational

Remark: Generally speaking, we have

$$\mathbf{F} \text{ conservative} \quad \Longrightarrow \quad \mathbf{F} \text{ irrotational}$$

Example of vector field with rotation

Vector field: **F** defined by

$$\mathbf{F} = \langle y, -x \rangle$$

Curl of **F**: We get

$$\mathrm{Curl}(\mathbf{F}) = g_x - f_y = -2$$

Interpretation:

F has a rotational component

Types of curves

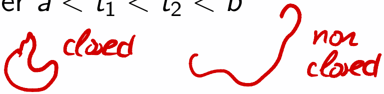
Definition 11.

Let

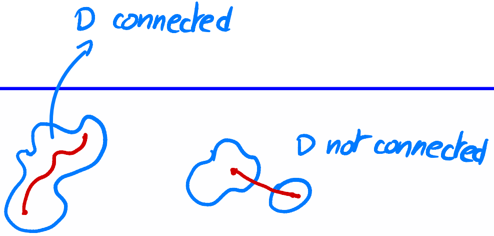
- Curve $C : [a, b] \to \mathbb{R}^2$
- C given as $\mathbf{r}(t)$

Then

1. C is a simple curve if
   $\hookrightarrow r(t_1) \neq r(t_2)$ whenever $a < t_1 < t_2 < b$
2. C is a closed curve if
   $\hookrightarrow r(a) = r(b)$



Types of domains



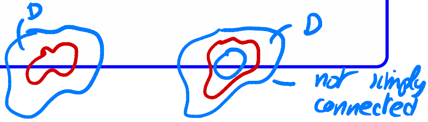
Definition 12.

Let

- D domain of $\mathbb{R}^2$

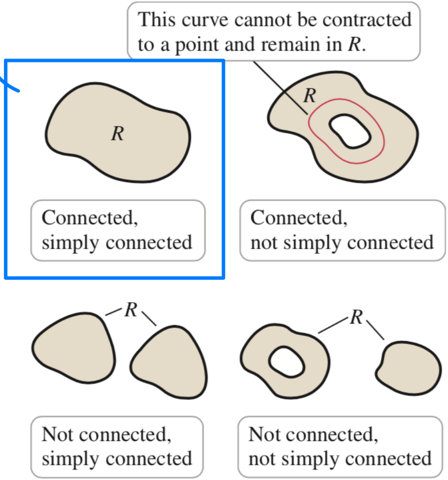
Then

1. D is a connected domain if
   $\hookrightarrow$ it is possible to connect any two points of D by a continuous curve lying in D
2. C is a simply connected domain if (no hole)
   $\hookrightarrow$ every closed simple curve in D can be deformed and contracted to a point in D



Connected and simply connected domains

most important

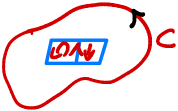


General assumptions

Hypothesis for this section:

- All curves C are closed and simple
  $\hookrightarrow$ In counterclockwise direction
- All domains R are connected and simply connected

Green's theorem



Theorem 13.

Let

- $\mathbf{F} = \langle f, g \rangle$ vector field in $\mathbb{R}^2$
- C simple closed curve, counterclockwise
- C delimits a region R

Then we have

1-d intg 2d-intg

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int\int_R \mathrm{Curl}(\mathbf{F})\,\mathrm{d}A \qquad (1)$$

George Green

Some facts about Green:

- Lifespan: 1793-1841, in England
- Self taught in Math, originally a baker
- Mathematician, Physicist
- 1st mathematical theory of electromagnetism
- Went to college when he was 40
- Died 1 year later (alcoholism?)

Interpretation of Green's theorem

Interpretation of the integral on C:

- $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is a circulation integral along the boundary C
- It accumulates the component of $\mathbf{F}$ tangential to $\mathbf{r}$

Interpretation of the integral on R:

- $\int\int_R \mathrm{Curl}(\mathbf{F})\, \mathrm{d}A$ accumulates rotation of $\mathbf{F}$ in R

Interpretation of the identity: Some cancellations occur
$\hookrightarrow$ the surface integral is reduced to a curve integral