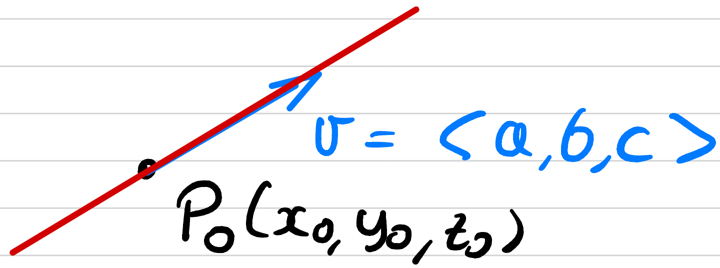
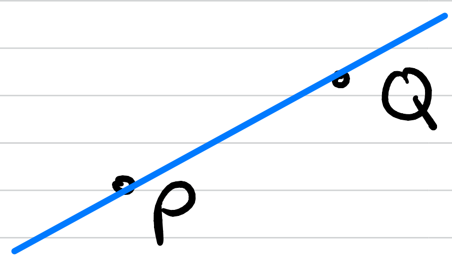
Summary , Wednesday session

Line : eq with given P_0 and $\vec{v}$



Eq : $\langle x, y, z \rangle = \langle x_0 + ta,\ y_0 + tb,\ z_0 + tc \rangle$

Line ver 2 : given P and Q

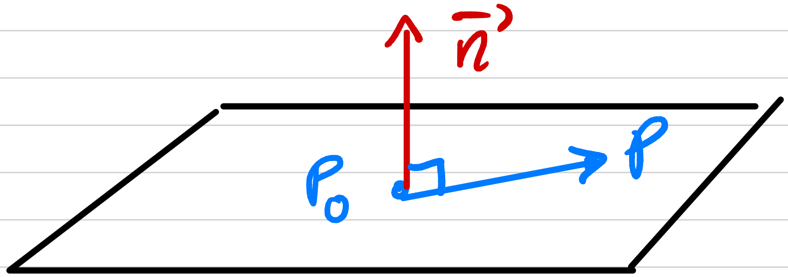


Take $P_0 = P$

and $\vec{v} = \vec{PQ}$

Plane ver 1

Given: P_0 and $\vec{n}$ normal



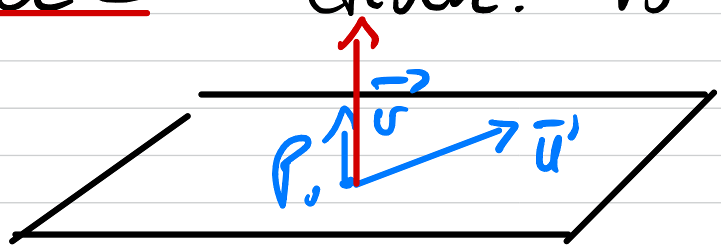
$n = \langle a, b, c \rangle$

Eq: $\overrightarrow{P_0P} \cdot \vec{n} = 0$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Plane ver 2

Given: P_0 and $\vec{u}, \vec{v}$ in the plane



Then compute

$\vec{u} \times \vec{v} = \vec{n}$

we are then back to ver. 1.

Intersecting planes (1)

Problem: Find an equation of the line of intersection of the planes

$$Q : x + 2y + z = 5$$

and

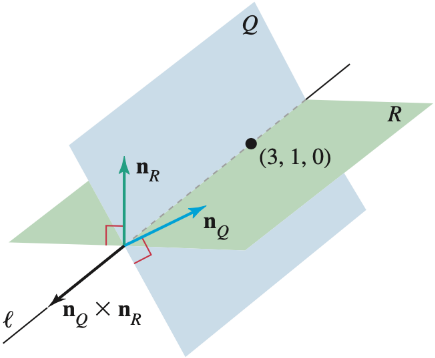
$$R : 2x + y - z = 7$$

Aim: Find P_0 and $\vec{v}$
⇒ Back to version 1

Strategy:

1. Find a point P_0 in $Q \cap R$
   ↪ Solve system
2. Find the direction $\mathbf{v}$ of $Q \cap R$
   ↪ Given by $\mathbf{v} = \mathbf{n}_Q \times \mathbf{n}_R$

Intersecting planes (2)



$\mathbf{n}_Q \times \mathbf{n}_R$ is a vector perpendicular to $\mathbf{n}_Q$ and $\mathbf{n}_R$.
Line ℓ is perpendicular to $\mathbf{n}_Q$ and $\mathbf{n}_R$.
Therefore, ℓ and $\mathbf{n}_Q \times \mathbf{n}_R$ are parallel to each other.

Intersection of Q and R

Q : $x + 2y + z = 5$ R: $2x + y - z = 7$

Step 1 : Find P_0. Set $z = 0$ in the eq for Q and R. We get a system

$$\begin{cases} x + 2y = 5 \\ 2x + y = 7 \end{cases}$$

Here unique solution: $x = 3$, $y = 1$

We get $\boxed{P_0(3, 1, 0)}$

Step 2: We take $\vec{v} = \vec{n}_Q \times \vec{n}_R$

Thus $\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \langle -3, 3, -3 \rangle$

Rmk Any multiple of $\langle -3, +3, -3 \rangle$ works.

Take $\boxed{\vec{v} = \langle 1, -1, 1 \rangle}$

Eq for the line: $P_0(3,1,0)$

$\vec{v} = \langle 1, -1, 1 \rangle$

Thus

$$\boxed{\langle x, y, z \rangle = \langle 3+t, 1-t, t \rangle}, \quad t \in \mathbb{R}$$

Intersecting planes (3)

System to find P_0 Take (e.g) $z = 0$. Then we get

$$x + 2y = 5, \qquad 2x + y = 7$$

Intersection: We find

$$P_0(3, 1, 0)$$

Intersecting planes (4)

Direction of the line: We have

$$\mathbf{n}_Q \times \mathbf{n}_R = \langle -3, 3, -3 \rangle$$

Thus we can take

$$\mathbf{v} = \langle 1, -1, 1 \rangle$$

Equation of the line:

$$\langle x, y, z \rangle = \langle 3 + t, 1 - t, t \rangle , \qquad t \in \mathbb{R}.$$

Outline

Cylinder

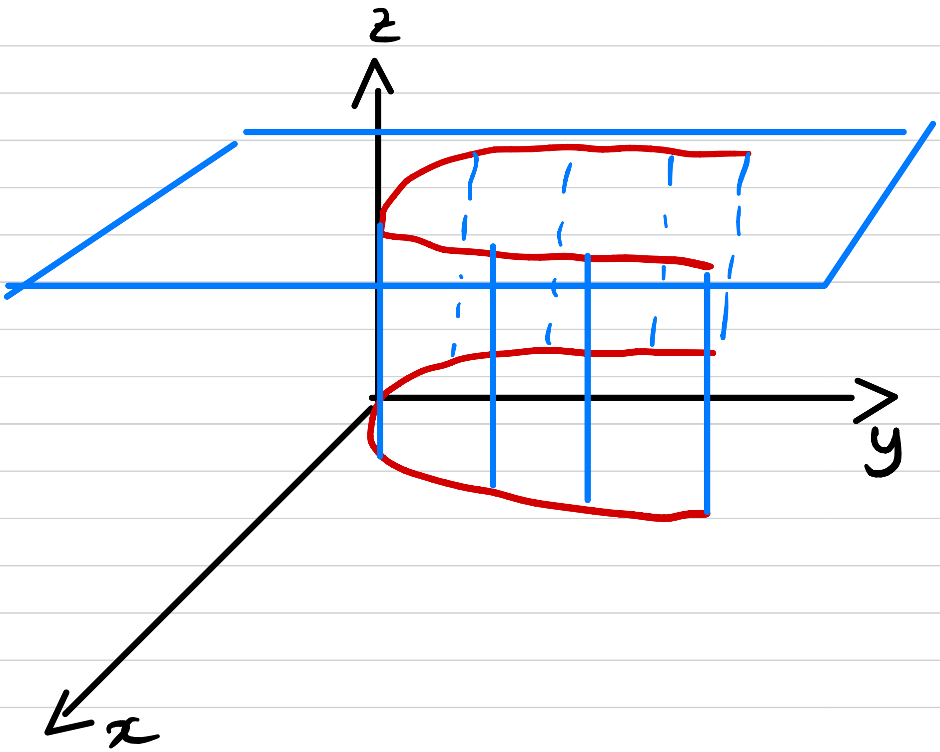
Shapes in $\mathbb{R}^3$:
Surfaces S whose equation contain the 3 variables x, y, z

Free variable: If a variable is missing from the equation of S
$\hookrightarrow$ It can take any value in $\mathbb{R}$ and is called free

Cylinder: Surface S with a free variable

Cylinder $y = x^2$ in $\mathbb{R}^3$

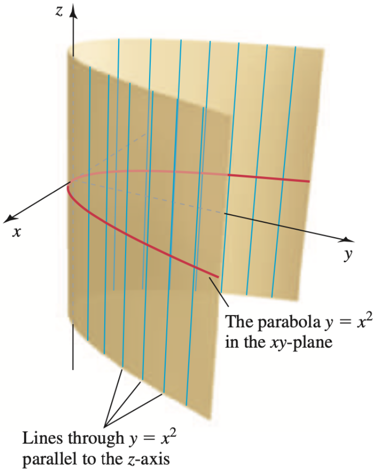
Example of cut : Take $z = 5$. Then for $z = 5$, the eq is still $y = x^2$



For $z = 0$, same thing.

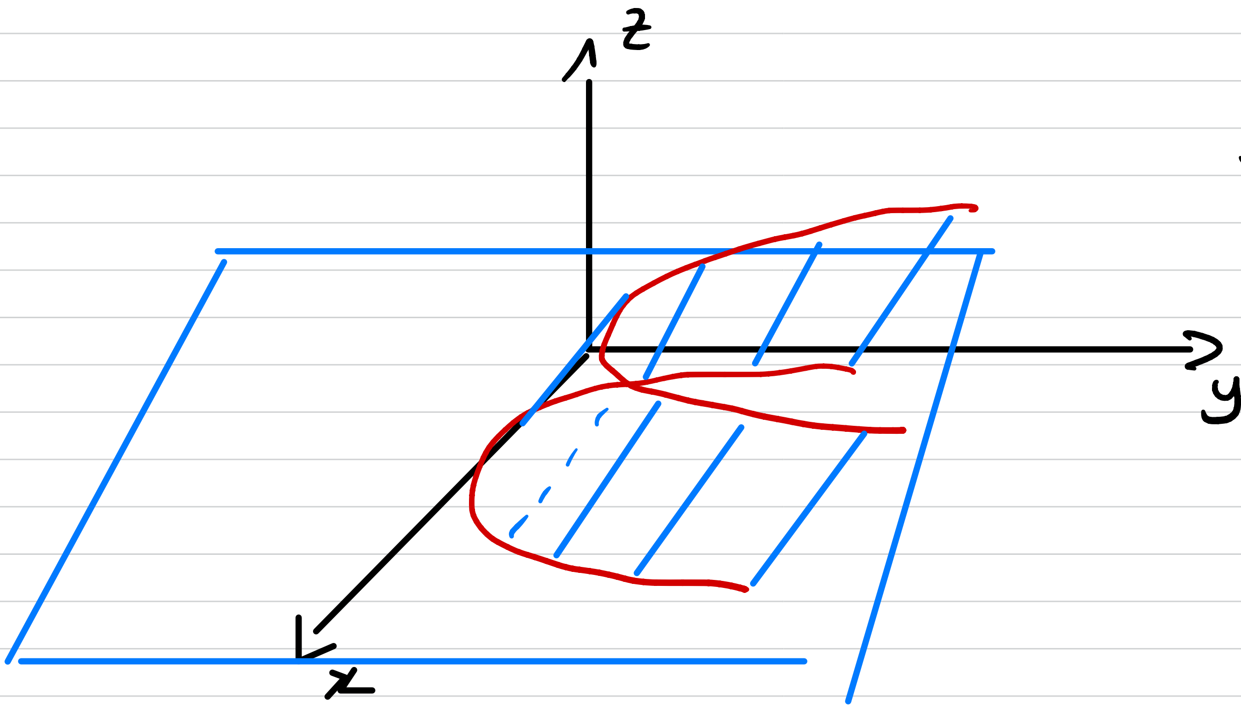
Example 1 of cylinder

Equation: $y = x^2$



Cylinder $y = z^2$. Free variable is x

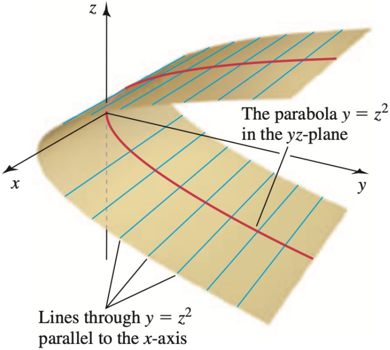
For $x = 6$, the eq is still $y = z^2$



Same thing for $x = 0$

Example 2 of cylinder

Equation: $y = z^2$



Trace

Definition 10.

Let

- S a surface in $\mathbb{R}^3$

Then

1. A trace of S is the set of points at which S intersects a plane that is parallel to one of the coordinate planes.
2. The traces in the coordinate planes are called the xy-trace, the xz-trace, and the yz-trace

Elliptic paraboloid (1)

Problem: Graph the surface

$$z = \frac{x^2}{16} + \frac{y^2}{4}$$

Traces:

- xy-trace: ellipse, whenever $z_0 \geq 0$
- xz-trace: parabola
- yz-trace: parabola

Surface $S: \frac{x^2}{16} + \frac{y^2}{4} = z$

xy-Trace: Fix $z = z_0$. The equation becomes

$$\frac{x^2}{16} + \frac{y^2}{4} = z_0 \longrightarrow \text{ellipse if } z_0 \geq 0$$

xz-Trace: Fix $y_0 = y$.

$$z = \frac{x^2}{16} + \frac{y_0^2}{4} \longrightarrow \text{parabola}$$

Elliptic paraboloid (2)