Outline

The main theorem

Theorem 23.

Consider

Convention: $\vec{n}$ points outside

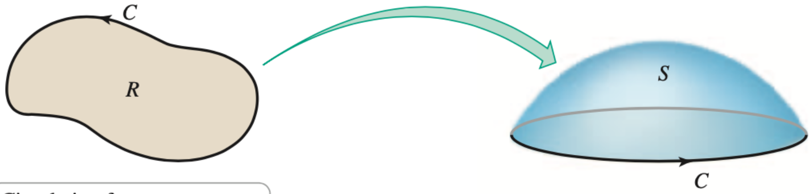
- An oriented surface S in $\mathbb{R}^3$
- S has a smooth boundary C
- $\mathbf{F} = \langle f, g, h \rangle$ vector field in $\mathbb{R}^3$
- $\mathrm{Curl}(\mathbf{F}) = \nabla \times \mathbf{F}$

Then we have

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S$$

From $\underbrace{\text{Green}}_{2\text{-d}}$ to $\underbrace{\text{Stokes}}_{3\text{-d}}$

From 2-d to 3-d:



Circulation form of Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \underbrace{(\nabla \times \mathbf{F}) \cdot \mathbf{k}}_{g_x - f_y} \, dA$$

Stokes' Theorem:

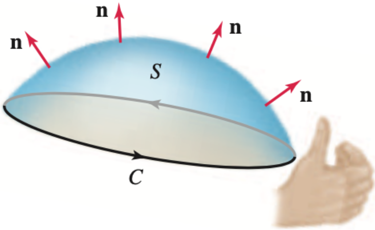
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$$

Orientations $\vec{n}$ pointing outward

Compatibility of orientations: Stokes' theorem involves

- An oriented surface
- An oriented curve (counterclockwise)

The orientations have to be compatible through the right hand rule



Verifying Stokes theorem (1)

Vector field:

$$\mathbf{F} = \langle z - y, x, -x \rangle$$

Surface: Hemisphere

$$S: \quad x^2 + y^2 + z^2 = 4 \quad \bigcap \quad \{z \geq 0\}$$

Corresponding curve: In xy-plane, circle oriented counterclockwise

$$C: \quad x^2 + y^2 = 4$$

Problem:
Verify Stokes' theorem in this context)

- Compute line integral
- Compute surface integral
- Check that they are =

$\vec{F} = \langle z-y,\ x,\ -x \rangle$

Curve $\quad C: \quad \vec{r}(t) = 2 \langle \underset{x}{\cos(t)},\ \underset{y}{\sin(t)},\ \underset{z}{0} \rangle$

$0 \leq t \leq 2\pi$

Derivative $\quad \vec{r}\,'(t) = 2 \langle -\sin(t),\ \cos(t),\ 0 \rangle$

Line integral

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle \overset{z}{0} \overset{-}{-} \overset{y}{2\sin(t)},\ \overset{x}{2\cos(t)},\ \overset{-x}{-2\cos(t)} \rangle$$

$$\cdot 2 \langle -\sin(t),\ \cos(t),\ 0 \rangle$$

$$= 4 \int_0^{2\pi} (\sin^2(t) + \cos^2(t) + 0)\, dt$$

$$= 4 \int_0^{2\pi} dt$$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = 8\pi}$$

Surface We will use spherical coordinates,

for which $\rho = 2$, $0 \le \varphi \le \frac{\pi}{2}$, $0 \le \theta \le 2\pi$

we set

$u = \varphi$, $v = \theta$

Then we get

$\langle x, y, z \rangle$ $\quad 0 \le u \le \frac{\pi}{2} \quad 0 \le v \le 2\pi$

$= \langle 2 \sin(u) \cos(v), 2 \sin(u) \sin(v), 2 \cos(u) \rangle$

$S: \langle 2\sin(u)\cos(v),\ 2\sin(u)\sin(v),\ 2\cos(u) \rangle$

$\vec{t}_u = \langle 2\cos(u)\cos(v),\ 2\cos(u)\sin(v),\ -2\sin(u) \rangle$

$\vec{t}_v = \langle -2\sin(u)\sin(v),\ 2\sin(u)\cos(v),\ 0 \rangle$

$\vec{t}_u \times \vec{t}_v$

$$\left| \begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 2\cos(u)\cos(v) & 2\cos(u)\sin(v) & -2\sin(u) & 2\cos(u)\cos(v) & 2\cos(u)\sin(v) \\ -2\sin(u)\sin(v) & 2\sin(u)\cos(v) & 0 & -2\sin(u)\sin(v) & 2\sin(u)\cos(v) \end{array} \right.$$

$\vec{i} \quad 0 \quad - \dots$

$\vec{j} \quad 4\sin^2(u)\sin(v) \quad - \dots \qquad \dots$

$\vec{k} \quad 4\cos(u)\sin(u)\cos^2(v) + 4\sin(u)\cos(u)\sin^2(v)$

we find

$$\boxed{\bar{r}'_u \times \bar{r}'_v = \langle 4\sin^2(u)\cos(v),\ 4\sin^2(u)\sin(v),\ 4\cos(u)\sin(u)\rangle}$$

Curl $\vec{F}$

$F = \langle z-y, x, -x \rangle$

$$\operatorname{Curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ z-y & x & -x \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ \partial_x & \partial_y \\ z-y & x \end{matrix}$$

$$= \begin{matrix} \vec{i}\,(0 + 0) \\ \vec{j}\,(1 + 1) \\ \vec{k}\,(1 + 1) \end{matrix}$$

$$\boxed{\operatorname{Curl}(\vec{F}) = \langle 0, 2, 2 \rangle}$$

Thus $\vec{F} = \vec{a} \times \langle x, y, z \rangle$, with

$\vec{a} = \langle 0, 1, 1 \rangle$ axis of rotation

$$\boxed{\vec{t}_u \times \vec{t}_v = \langle 4\sin^2(u)\cos(v),\ 4\sin^2(u)\sin(v),\ 4\cos(u)\sin(u)\rangle}$$

$$\boxed{\mathrm{Curl}(\vec{F}) = \langle 0, 2, 2\rangle}$$

$0 \le u \le \pi/2 \qquad 0 \le v \le 2\pi$

Rhs of Stokes $\iint_S \mathrm{Curl}(\vec{F}) \cdot \vec{n}\, dS$

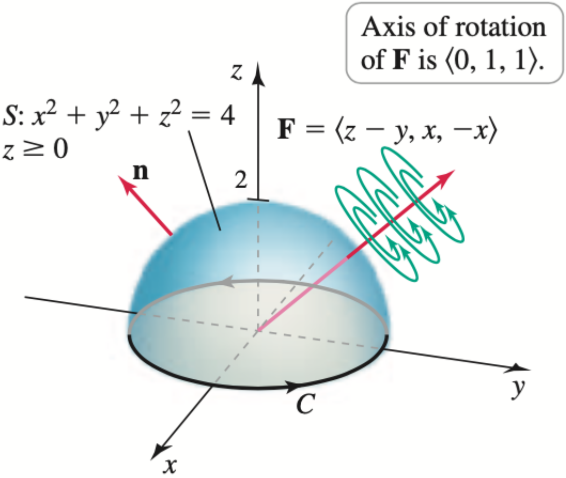
$$= \int_0^{\pi/2}\int_0^{2\pi} \langle 4\sin^2(u)\cos(v),\ 4\sin^2(u)\sin(v),\ 4\cos(u)\sin(u)\rangle \cdot \langle 0,2,2\rangle\, dv\, du$$

$$= 8\int_0^{\pi/2}\int_0^{2\pi} \left(\sin^2(u)\sin(v) + \sin(u)\cos(u)\right) dv\, du$$

$$= 8\left\{ \int_0^{\pi/2} \sin^2(u)\, du \underbrace{\int_0^{2\pi} \sin(v)\, dv}_{=0} + 2\pi \int_0^{\pi/2} \underbrace{\sin(u)\cos(u)}_{\frac{1}{2}\sin(2u)}\, du \right\} = 8\pi$$

Conclusion: $\iint_S \mathrm{Curl}(\vec{F}) \cdot \vec{n}\, dS = \int_C \vec{F} \cdot d\vec{r}$

Verifying Stokes theorem (2)



Verifying Stokes theorem (3)

Parametric equation for C:

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 0\rangle$$

Parametric equation for $\mathbf{F}$: Along C we have

$$\mathbf{F} = \langle z - y, x, -x\rangle = 2\,\langle -\sin(t), \cos(t), -\cos(t)\rangle$$

Dot product: We have

$$\mathbf{F}(t) \cdot \mathbf{r}'(t) = \left(\cos^2(t) + \sin^2(t)\right) = 4$$

Verifying Stokes theorem (4)

Line integral:

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s &= \oint_C \mathbf{F}(t)\cdot\mathbf{r}'(t)\,\mathrm{d}t \\ &= 4\int_0^{2\pi} \mathrm{d}t\end{aligned}$$

Thus we get

$$\oint_C \mathbf{F}(t)\cdot\mathbf{r}'(t)\,\mathrm{d}t = 8\pi$$

Verifying Stokes theorem (5)

Expression for $\mathrm{Curl}(\mathbf{F})$: We have

$$\mathrm{Curl}(\mathbf{F}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & x & -x \end{vmatrix}$$

Computation: We find that $\mathbf{F}$ is a rotation with axis $\langle 0,1,1 \rangle$

$$\mathrm{Curl}(\mathbf{F}) = \langle 0,2,2 \rangle$$

Verifying Stokes theorem (6)

Parametrization of S: We take

$$\mathbf{r}(u,v) = \langle 2\sin(u)\cos(v),\, 2\sin(u)\sin(v), 2\cos(u)\rangle\,, \quad (u,v)\in R,$$

with

$$R = \{0 \le u \le \pi/2,\, 0 \le v \le 2\pi\}$$

Verifying Stokes theorem (7)

Normal vector: We have

$$\begin{aligned} \mathbf{t}_u &= \langle 2\cos(u)\cos(v),\ 2\cos(u)\sin(v), -2\sin(u)\rangle\,, \\ \mathbf{t}_v &= \langle -2\sin(u)\sin(v),\ 2\sin(u)\cos(v), 0\rangle\,, \end{aligned}$$

Thus

$$\mathbf{t}_u \times \mathbf{t}_v = \left\langle 4\sin^2(u)\cos(v),\ 4\sin^2(u)\sin(v), 4\cos(u)\sin(u)\right\rangle$$

Verifying Stokes theorem (8)

Surface integral: We get

$$\int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S = \int\int_R \mathrm{Curl}(\mathbf{F}) \cdot (\mathbf{t}_u \times \mathbf{t}_v) \, \mathrm{d}A$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \langle 0, 2, 2 \rangle \cdot \left\langle 4\sin^2(u)\cos(v),\ 4\sin^2(u)\sin(v), 4\cos(u)\sin(u) \right\rangle \, \mathrm{d}u\mathrm{d}v$$

$$= 8 \int_0^{2\pi} \int_0^{\pi/2} \left(\sin^2(u)\sin(v) + \sin(u)\cos(u) \right) \, \mathrm{d}u\mathrm{d}v$$

$$= 8 \int_0^{2\pi} \int_0^{\pi/2} sin(u)\cos(u) \, \mathrm{d}u\mathrm{d}v$$

We get a positive flux (since **F** points outward):

$$\int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S = 8\pi$$

Verifying Stokes theorem (9)

Verification: We have found

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S = 8\pi$$