Stokes theorem

C: counterclockwise

S: $\vec{n}$ pointing outward

Then

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{Curl}(\vec{F}) \cdot \vec{n} \, dS}$$

Stokes theorem for a line integral (1)

Vector field:

$$\mathbf{F} = \left\langle z, -z, x^2 - y^2 \right\rangle$$

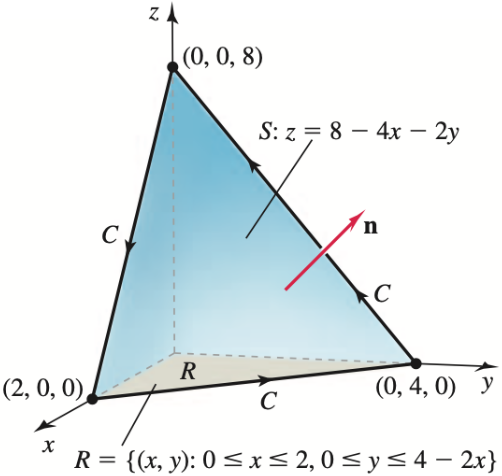
Surface: Plane in the first octant, with **n** pointing upward

$$S: \quad z = 8 - 4x - 2y \quad \bigcap \quad \{x \geq 0,\ y \geq 0,\ z \geq 0\}$$

Corresponding curve:
Three lines delimiting $S \longrightarrow$ Long to parametrize

Problem: In order to avoid a parametrization of C
$\hookrightarrow$ Evaluate $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ as a surface integral

Stokes theorem for a line integral (2)



Vector field $\vec{F} = \langle z, -z, x^2 - y^2 \rangle$

Aim We wish to compute $\int_C \vec{F} \cdot d\vec{r}$

It will be slightly easier to compute

$$\iint_S \operatorname{Curl}(\vec{F}) \cdot \vec{n} \, dS \quad \left(= \int_C \vec{F} \cdot d\vec{r} \right) \quad \text{(Stokes)}$$

vecta field $\vec{F} = \langle z, -z, x^2-y^2 \rangle$

curl $\quad \text{Curl}(\vec{F})$

$$= \left| \begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \partial_x & \partial_y & \partial_z & \partial_x & \partial_y \\ z & -z & x^2-y^2 & z & -z \end{array} \right.$$

$$= \vec{i}\,(-2y + 1)$$

$$+ \vec{j}\,(1 - 2x)$$

$$+ \vec{k}\,(0 - 0)$$

$$\boxed{\text{Curl}(\vec{F}) = \langle 1-2y,\ 1-2x,\ 0 \rangle}$$

$$\boxed{\text{Curl}(\vec{F}) = \langle \overbrace{1-2y}^{f}, \overbrace{1-2x}^{g}, \overbrace{0}^{h} \rangle}$$

<u>Surface</u> z is an explicit function of x, y:

$$z = 8 - 4x - 2y \quad \begin{cases} z_x = -4 \\ z_y = -2 \end{cases}$$

Domain of integration: $8 - 4x \geq 0 \iff \begin{cases} x \geq 0 \\ x \leq 2 \end{cases}$

Then $8 - 4x - 2y \geq 0 \iff \begin{cases} y \geq 0 \\ y \leq 4 - 2x \end{cases}$

<u>Surface integral</u>

$$\iint_S \vec{F} \cdot \vec{n} \, dS$$

$$= \int_0^2 \int_0^{4-2x} \{-(1-2y)\cdot(-4) - (1-2x)(-2) + 0\} \, dy \, dx$$

$$= \int_0^2 \int_0^{4-2x} \{6 - 4x - 8y\} \, dy \, dx$$

$$= \int_0^2 \underbrace{6y - 4xy - 4y^2 \Big|_0^{4-2x}}_{\text{polynomial in } x} = \cdots = -\frac{88}{3}$$

Conclusion

$$\int_C \vec{F} \cdot d\vec{r} = -\frac{88}{3}$$ (clockwise global rotation of $\vec{F}$ along C)

Check

Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ and check that

$$\int_C \vec{F} \cdot d\vec{r} = -\frac{88}{3}$$

Stokes theorem for a line integral (3)

Expression for $\mathrm{Curl}(\mathbf{F})$: We have

$$\mathrm{Curl}(\mathbf{F}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -z & x^2 - y^2 \end{vmatrix}$$

Computation: We find

$$\mathrm{Curl}(\mathbf{F}) = \langle 1 - 2y, \, 1 - 2x, \, 0 \rangle$$

Stokes theorem for a line integral (4)

Parametrization of S: We take the explicit version

$$z = 8 - 4x - 2y, \quad (x, y) \in R,$$

with

$$R = \{0 \le x \le 2,\, 0 \le y \le 4 - 2x\}$$

Stokes theorem for a line integral (5)

Normal vector: We write the plane as

$$4x + 2y + z = 8$$

Thus

$$\mathbf{n} = \langle 4, 2, 1 \rangle$$

Formula used for the surface integral: Explicit case in Definition 22

$$\int\int_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \int\int_R (-f \, z_x - g \, z_y + h) \, \mathrm{d}A$$

Stokes theorem for a line integral (6)

Surface integral: We get

$$\begin{aligned}
&\int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n}\, \mathrm{d}S \\
&= \int_0^2 \int_0^{4-2x} \langle 4, 2, 1 \rangle \cdot \langle 1 - 2y,\, 1 - 2x,\, 0 \rangle \, \mathrm{d}x\mathrm{d}y \\
&= \int_0^2 \int_0^{4-2x} (6 - 4x - 8y) \, \mathrm{d}x\mathrm{d}y
\end{aligned}$$

We obtain:

$$\int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n}\, \mathrm{d}S = -\frac{88}{3}$$

Stokes theorem for a line integral (7)

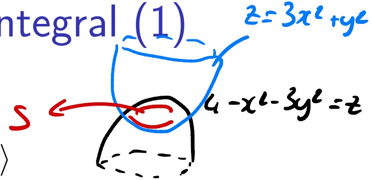
Computation of the line integral: We have

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n}\, \mathrm{d}S = -\frac{88}{3}$$

Remark:

We get a negative flux (circulation is going clockwise)

Stokes theorem for a surface integral (1)



Vector field:

$$\mathbf{F} = \langle -y, x, z \rangle$$

Surface: Part of a paraboloid within another paraboloid

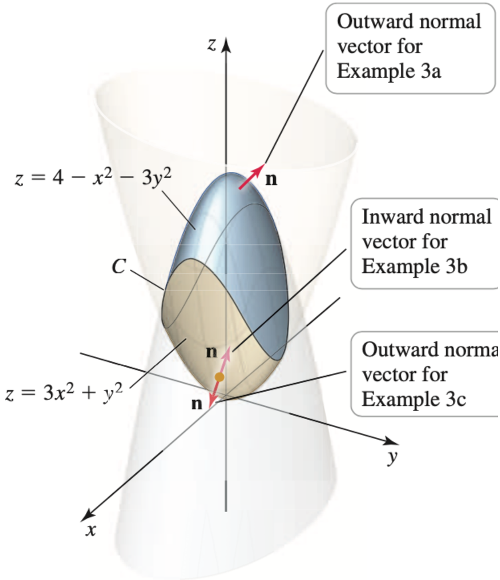
$$S: \quad z = 4 - x^2 - 3y^2 \quad \bigcap \quad \left\{ z \geq 3x^2 + y^2 \right\},$$

with $\mathbf{n}$ pointing upward

Corresponding curve:
Intersection of the 2 paraboloids

Problem: In order to avoid a parametrization of S
$\hookrightarrow$ Evaluate $\int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S$ as a line integral

Stokes theorem for a surface integral (2)



Curve: If $z = 4 - x^2 - 3y^2 = 3x^2 + y^2$ → intersection of 2 paraboloids

we get $4x^2 + 4y^2 = 4$

$\Rightarrow \quad x^2 + y^2 = 1$ (unit circle in xy-plane)

Parametrization

$C: \{ \vec{r}(t) = \langle \cos(t), \sin(t), 3\cos^2(t) + \sin^2(t) \rangle; \; 0 \leq t \leq 2\pi \}$

$\vec{r}'(t) = \langle -\sin(t), \cos(t), -6\sin(t)\cos(t) + 2\cos(t)\sin(t) \rangle$

$\vec{r}' = \langle -\sin(t), \cos(t), -4\sin(t)\cos(t) \rangle$

$$\vec{r}\,' = \langle -\sin(t), \cos(t), -4\sin(t)\cos(t) \rangle$$

Line integral

$$\int_C \vec{F}\, d\vec{r}$$

$$= \int_0^{2\pi} \langle -\sin(t), \cos(t), 3\cos^2(t) + \sin^2(t) \rangle$$

$$\cdot \langle -\sin(t), \cos(t), -4\sin(t)\cos(t) \rangle \, dt$$

$$= \cdots = 2\pi$$