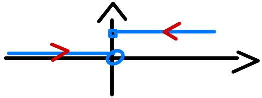
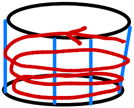
Limits and continuity (1)





Function: We define

located on a cylinder $\Leftarrow$ $\mathbf{r}(t) = \left\langle \cos(\pi\, t), \sin(\pi\, t), e^{-t} \right\rangle$

$\hookrightarrow 0$ as $t \to \infty$

Questions:

by continuity

1. Graph $\mathbf{r}$
2. Evaluate $\lim_{t\to 2} \mathbf{r}(t)$ $= \vec{r}(2) = \langle 1, 0, e^{-2} \rangle$
3. Evaluate $\lim_{t\to\infty} \mathbf{r}(t)$ $\longrightarrow$ Does not exist (cos and sin are oscillating functions)
4. At what points is $\mathbf{r}$ continuous?

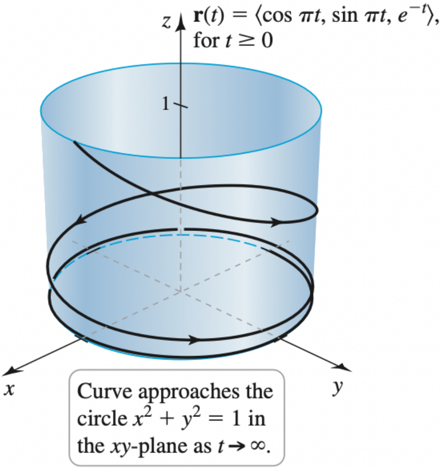
$\hookrightarrow$ Since $\cos(\pi t)$, $\sin(\pi t)$, e^{-t} are all continuous functions on $\mathbb{R}$, $\vec{r}(t)$ is also continuous on $\mathbb{R}$

Limits and continuity (2)

Answers

1. $\lim_{t \to 2} \mathbf{r}(t) = \langle 1, 0, e^{-2} \rangle$
2. No limit. As $t \to \infty$
   $\hookrightarrow$ $\mathbf{r}(t)$ approaches the unit circle in xy-plane
3. $\mathbf{r}$ is continuous everywhere

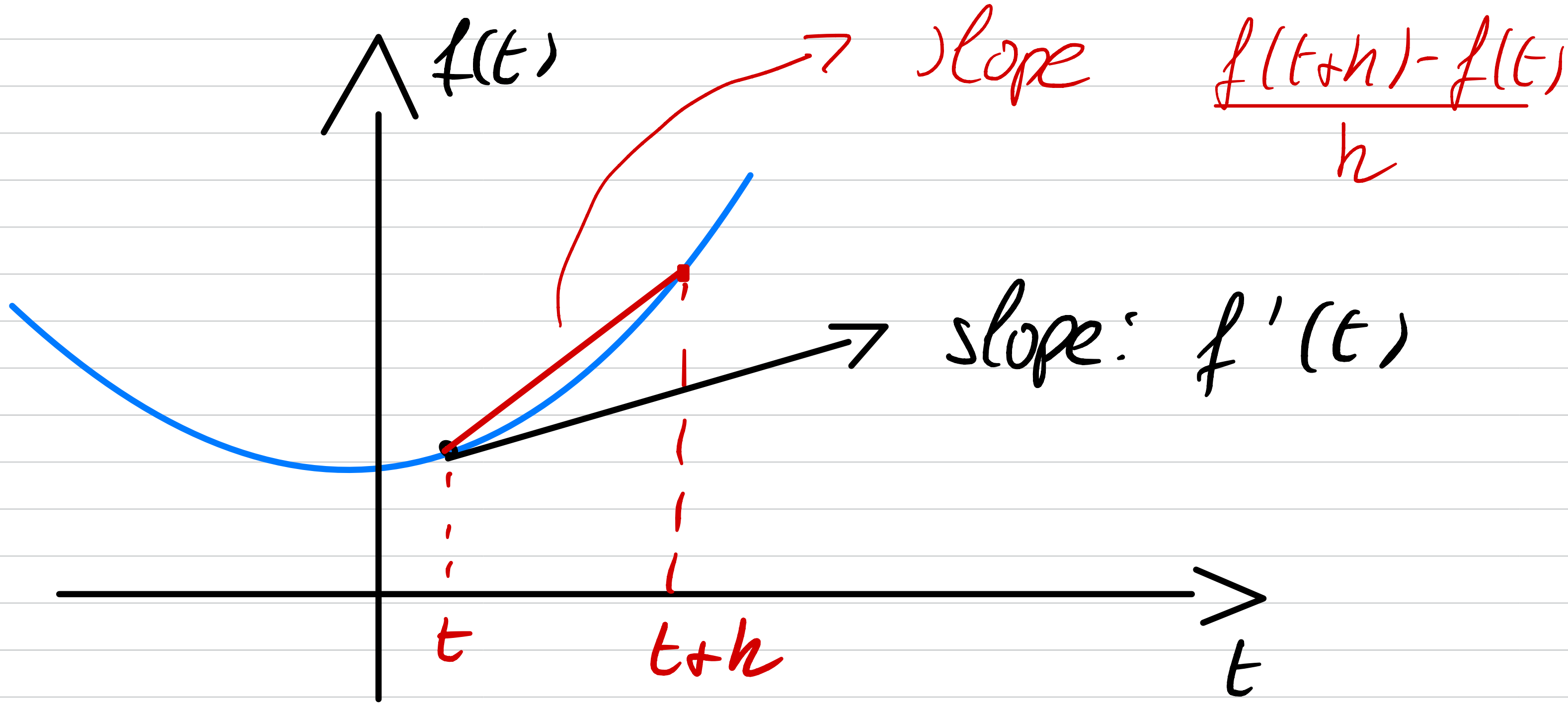
Limits and continuity (3)



Outline

1. Vector-valued functions
2. Calculus of vector-valued functions
3. Motion in space
4. Length of curves
5. Curvature and normal vector

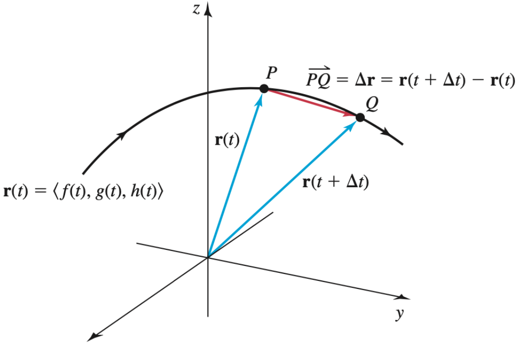
Interpretation of derivative for $f: \mathbb{R} \to \mathbb{R}$



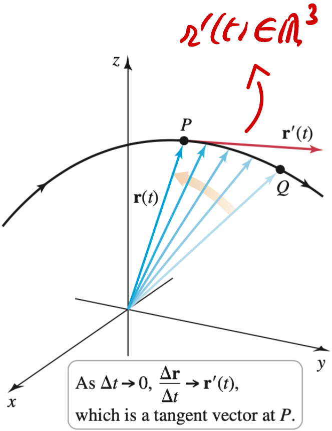
Another point of view: we have a curve in $\mathbb{R}^2$, of the form

$\vec{r}(t) = \langle t, f(t) \rangle \Rightarrow \vec{r}\,'(t) = \langle 1, f'(t) \rangle$

Derivative and velocity



(a)



(b)

Spiral on cone example

Function: Consider the curve defined by

$$\mathbf{r}(t) = \langle t\cos(t), t\sin(t), t\rangle$$

Derivative: We get

$$\mathbf{r}'(t) = \langle -t\sin(t) + \cos(t), t\cos(t) + \sin(t), 1\rangle$$

Related surface: $\mathbf{r}$ is a spiral on the cone

$$x^2 + y^2 = z^2$$

Curve $\vec{r}(t) = \langle \overbrace{t\cos(t)}^{x(t)}, \overbrace{t\sin(t)}^{y(t)}, \overbrace{t}^{z(t)} \rangle$

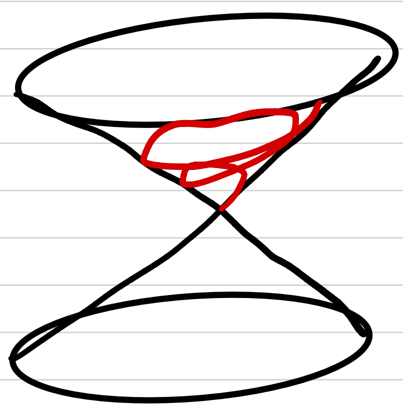
Graph: $x^2(t)+y^2(t) = t^2\cos^2(t) + t^2\sin^2(t)$

$$= t^2 = z^2$$

Thus $\vec{r}(t) \in$ cone $x^2+y^2 = z^2$

This cone is such that each xy-trace is a circle $x^2+y^2=z^2$

The radius of this circle grows like z



$$\boxed{\vec{r}(t) = \langle t\cos(t), t\sin(t), t \rangle}$$

Derivative

$$\vec{r}'(t) = \langle \cos(t) - t\sin(t), \sin(t) + t\cos(t), 1 \rangle$$

Unit tangent vector

Definition 2.

Let

- $\mathbf{r}(t)$ a vector-valued function
- Assume $\mathbf{r}'(t) \neq 0$

Then the unit tangent vector of $\mathbf{r}$ at time t is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Rmk By definition, $|\vec{T}(t)| = 1$

Spiral example We have computed

$$\vec{r}'(t) = \langle -t\sin(t)+\cos(t),\ t\cos(t)+\sin(t),\ 1\rangle$$

Thus

$$|\vec{r}'(t)|^2 = (-t\sin(t)+\cos(t))^2 + (t\cos(t)+\sin(t))^2 + 1^2$$

$$= t^2\sin^2(t) - 2t\sin(t)\cos(t) + \cos^2(t) + t^2\cos^2(t) + 2t\cos(t)\sin(t) + \sin^2(t) + 1$$

$$= t^2+1+1 = t^2+2$$

$$\Rightarrow \quad |\vec{r}'(t)| = \sqrt{t^2+2}$$

$$\boxed{\vec{T}(t) = \frac{\vec{r}'(t)}{\sqrt{t^2+2}}}$$

Going back to counterclockwise / clockwise

In $\mathbb{R}^2$ consider the curve

$$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$$

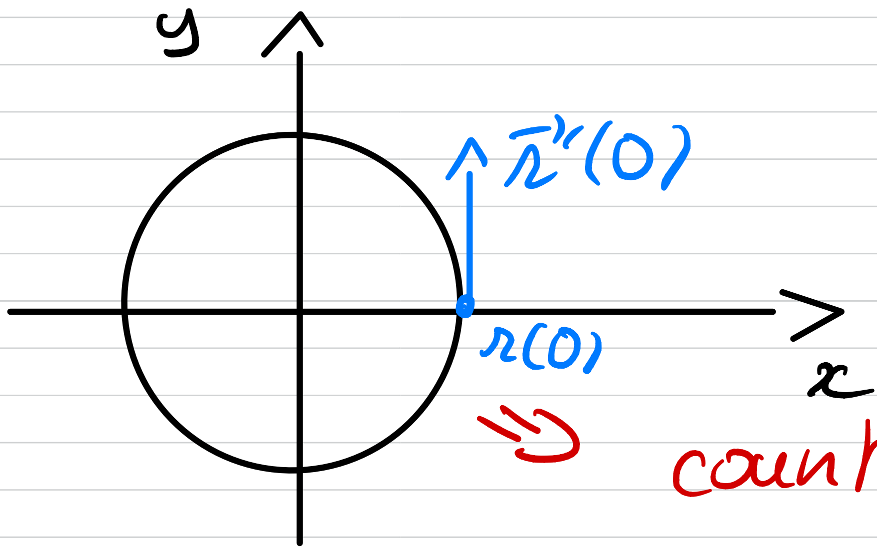
$\vec{r}'(0) = \langle 1, 0 \rangle$

$\vec{r}'(0) = \langle 0, 1 \rangle$

$\Rightarrow$

We compute

$$\vec{r}'(t) = \langle -\sin(t), \cos(t) \rangle$$



$\Rightarrow$ counterclockwise

Spiral on cone example

Function: Consider the curve defined by

$$\mathbf{r}(t) = \langle t\cos(t), t\sin(t), t\rangle$$

Derivative: We have seen

$$\mathbf{r}'(t) = \langle -t\sin(t) + \cos(t), t\cos(t) + \sin(t), 1\rangle$$

Unit tangent: We get

$$\mathbf{T}(t) = \left\langle \frac{-t\sin(t) + \cos(t)}{\sqrt{t^2+2}}, \frac{t\cos(t) + \sin(t)}{\sqrt{t^2+2}}, \frac{1}{\sqrt{t^2+2}} \right\rangle$$

Product rule for $f, g : \mathbb{R} \longrightarrow \mathbb{R}$

$$\boxed{(fg)' = f'g + fg'}$$

Product rules

Theorem 3.

Let

- $\mathbf{u}, \mathbf{v}$ vector-valued functions
- f real-valued function

Then we have

$$\begin{aligned}
[f(t)\mathbf{u}(t)]' &= f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) \\
[\mathbf{u}(t) \cdot \mathbf{v}(t)]' &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \\
[\mathbf{u}(t) \times \mathbf{v}(t)]' &= \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)
\end{aligned}$$

Example of product rule

$$f(t)\,\vec{r}(t) = \langle e^t, te^t, t^2e^t\rangle = e^t\langle 1, t, t^2\rangle$$

$$(f(t)\,\vec{r}(t))' = \underbrace{e^t}_{f'}\langle \underbrace{1, t, t^2}_{r}\rangle + \underbrace{e^t}_{f}\langle \underbrace{0, 1, 2t}_{r'}\rangle$$

Functions: Consider

$$\mathbf{r}(t) = \left\langle 1, t, t^2 \right\rangle, \qquad f(t) = e^t$$

Product derivative: We find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f(t)\mathbf{r}(t)\right] = e^t\left\langle 1, t+1, t^2+2t \right\rangle$$

Antiderivative

Definition 4.

Consider

- $\mathbf{r}$ of the form $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$
- F, G, H antiderivatives of f, g, h respectively
- $\mathbf{R}(t) = \langle F(t), G(t), H(t) \rangle$

Then we have

$$\int \mathbf{r}(t)\,\mathrm{d}t = R(t) + \langle C_1, C_2, C_3 \rangle$$

Example of antiderivative

Function: Consider

$$\mathbf{r}(t) = \left\langle \frac{t}{\sqrt{t^2+2}}, e^{-3t}, \sin(4t)+1 \right\rangle$$

Antiderivative: We get

$$\int \mathbf{r}(t)\,\mathrm{d}t = \left\langle \sqrt{t^2+2}, -\frac{1}{3}e^{-3t}, t - \frac{1}{4}\cos(4t) \right\rangle + \mathbf{C}$$

$\boxed{\text{Curve}}$

$$\vec{r}'(t) = \left\langle \frac{t}{\sqrt{t^2+2}}, e^{-3t}, \sin(4t)+1 \right\rangle$$

Then

$= \frac{1}{2}(t^2+2)'$

$$\vec{R}(t) = \left\langle \int \frac{t}{\sqrt{t^2+2}}\,dt, \int e^{-3t}\,dt, \int \sin(4t)+1\,dt \right\rangle$$

$$= \left\langle \sqrt{t^2+2}+C_1, -\frac{1}{3}e^{-3t}+C_2, -\frac{1}{4}\cos(4t)+t+C_3 \right\rangle$$