Outline

Recalling functions of 1 variable (1)

Example of function:

$$y = f(x) = \sqrt{9 - x^2}$$

Questions:

1. Domain of f?
2. Range of f?

Function: $f(x) = \sqrt{9 - x^2}$

Domain: $\boxed{-3 \leq x \leq 3}$

Otherwise stated, we wish

$$9 - x^2 \geq 0 \iff x^2 \leq 9$$
$$\iff -3 \leq x \leq 3$$

Range: $[0, 3]$

Details: Large values of f if x^2 is small

Thus

$$\max f(x) = f(0) = \sqrt{9} = 3$$
$$\min f(x) = f(\pm 3) = \sqrt{9-9} = 0$$

Range: $\boxed{[0, 3]}$

Recalling functions of 1 variable (2)

Recalling the function:

$$y = f(x) = \sqrt{9 - x^2}$$

Domain:

$$x \in [-3, 3]$$

Range:

$$y \in [0, 3]$$

Functions of 2 variables: example (1)

Example of function:

$$z = f(x, y) = \sqrt{9 - x^2} - \sqrt{25 - y^2}$$

Questions:

1. Domain of f?
2. Range of f?

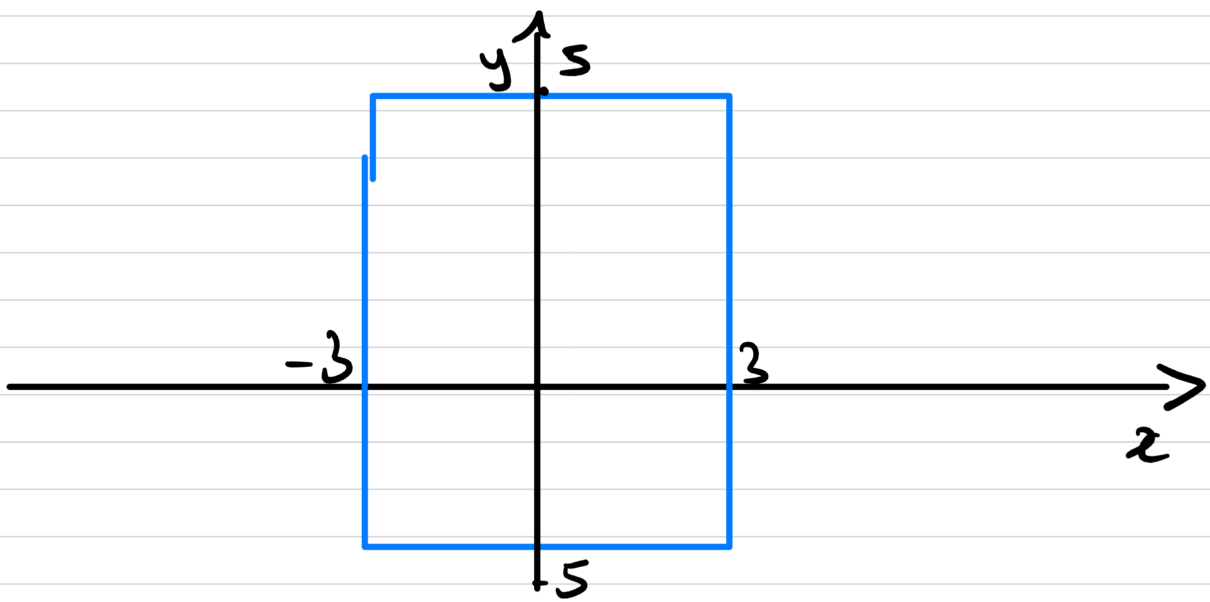
Function: $f(x,y) = \sqrt{9-x^2} - \sqrt{25-y^2}$

Domain: In order for f to be defined, we need

$9-x^2 \geq 0$ and $25-y^2 \geq 0$

$\Leftrightarrow -3 \leq x \leq 3$ $\quad -5 \leq y \leq 5$

Domain $[-3,3] \times [-5,5]$ (rectangle)



Function: $f(x,y) = \sqrt{9-x^2} - \sqrt{25-y^2}$

Range: $f = a - b$ with $a, b \geq 0$.

Thus f is large when a is large, b small

f " small " a is small, b large

Hence

a large ↑ ↗ b small

$\max f = f(0, \pm 5) = 3$

$\min f = f(\pm 3, 0) = -5$

Range = [-5, 3]

Functions of 2 variables: example (2)

Recalling the function:

$$z = f(x, y) = \sqrt{9 - x^2} - \sqrt{25 - y^2}$$

Domain:

$$(x, y) \in [-3, 3] \times [-5, 5]$$

Range: Looking at lines $x = \pm 3$ and $y = \pm 5$, we get

$$y \in [-5, 3]$$

Contour and level curves

Definition 1.

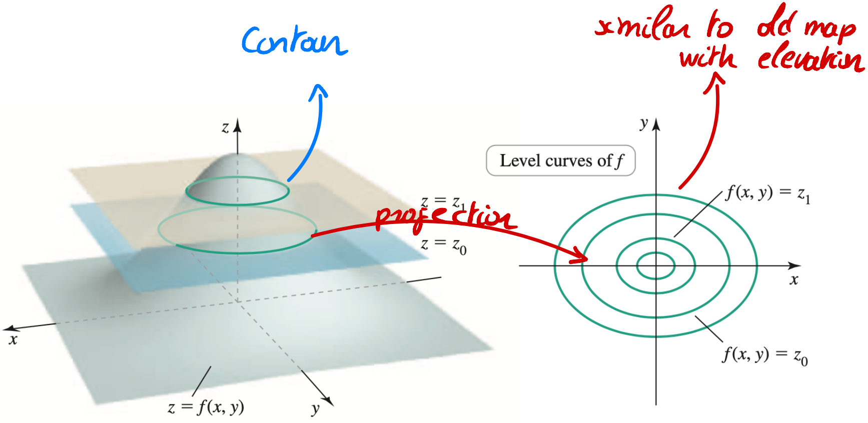
Contour curve:

Intersection of the surface $(x, y, f(x, y))$ and plane $z = z_0$

Level curve:

Projection of contour curve on xy-plane

Contour and level curves: illustration



Example of level curves (1)

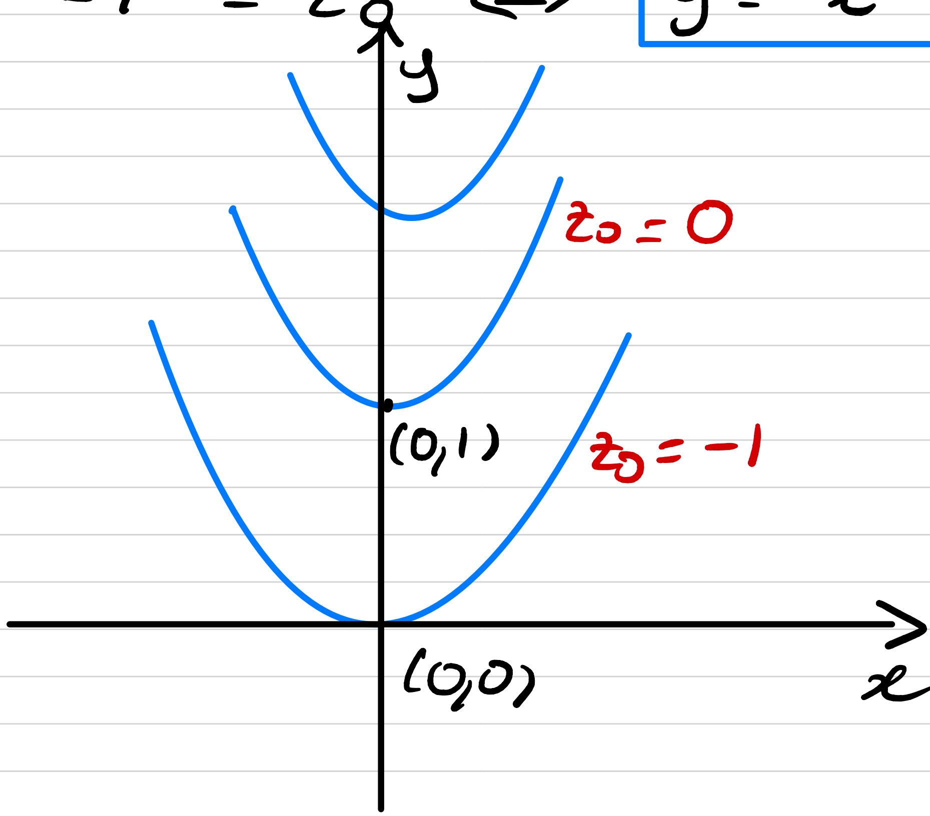
Function:

$$f(x, y) = y - x^2 - 1$$

<u>Function</u> : $f(x,y) = y - x^2 - 1$ (with $f(x,y) = z$)

<u>Generic level curve</u>: Fix $z = z_0$. We get an equation for (x,y).

$$y - x^2 - 1 = z_0 \iff \boxed{y = x^2 + 1 + z_0}$$ ← parabola



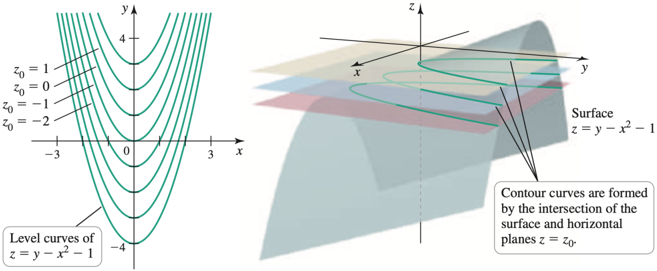
Example of level curves (2)

Function:

$$f(x, y) = y - x^2 - 1$$

Level curves: For $z_0 \in \mathbb{R}$, we get the parabola

$$y = x^2 + 1 + z_0$$



Example 2 of level curves (1)

Function:

$$f(x, y) = \exp\left(-x^2 - y^2\right)$$

Function : $f(x,y) = e^{-(x^2+y^2)}$

Domain : Defined for all $(x,y) \in \mathbb{R}^2$

Range : x^2+y^2 takes all values in $\mathbb{R}_+$

$-(x^2+y^2)$ " " " " $\mathbb{R}_-$

$e^{-(x^2+y^2)}$ " " " " $(0,1]$

Level curves : We should only consider

$$e^{-(x^2+y^2)} = z_0$$

for $z_0 \in (0,1]$

Eq. for a level curve. Consider $z_0 \in (0,1]$

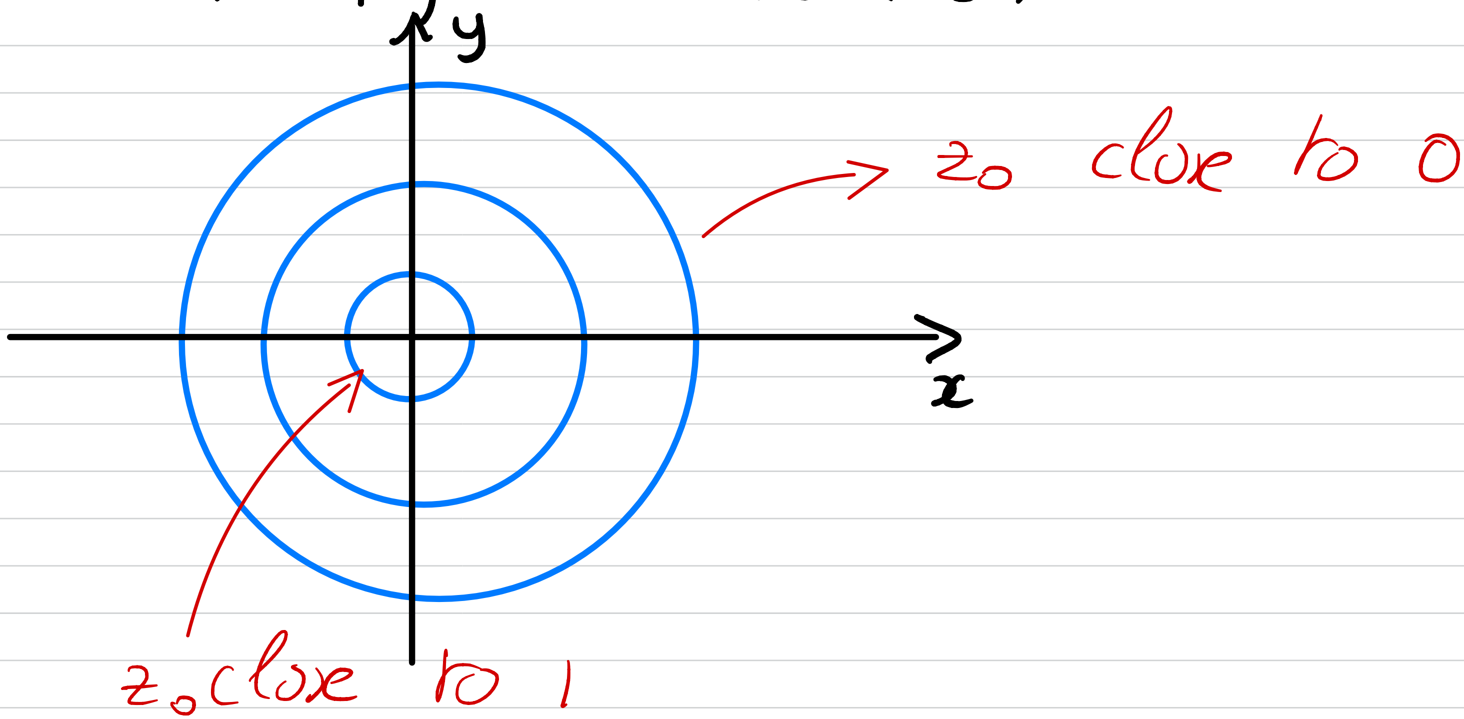
Then $f(x,y) = z_0$

$\iff e^{-(x^2+y^2)} = z_0$

log

$\iff -(x^2+y^2) = \ln(z_0)$

$\iff \overbrace{x^2+y^2}^{\geq 0} = \overbrace{-\ln(z_0)}^{\geq 0 \text{ since } 0 < z_0 \leq 1}$



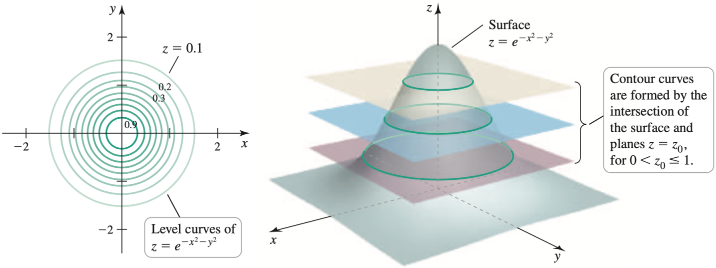
Example 2 of level curves (2)

Function:

$$f(x, y) = \exp\left(-x^2 - y^2\right)$$

Level curves: For $z_0 \in (0, 1]$, we get the circle

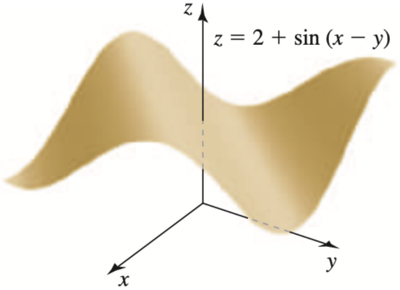
$$x^2 + y^2 = -\ln(z_0)$$



Example 3 of level curves (1)

Function:

$$f(x, y) = 2 + \sin(x - y)$$



Function : $f(x,y) = 2 + \sin(x-y)$

Range : $\sin \in [-1, 1]$

Thus $f(x,y) \in [1, 3]$ → Range

Level curve: For $z_0 \in [1, 3]$

$f(x,y) = z_0 \iff 2 + \sin(x-y) = z_0$

$\iff \boxed{\sin(x-y) = z_0 - 2}$

Take $z_0 = 2$. Then eq. becomes

$\sin(x-y) = 0 \iff x - y = k\pi$

$\iff y = x - k\pi, \ k \in \mathbb{Z}$

(family of lines)

Example 3 of level curves (2)

Function:

$$f(x, y) = 2 + \sin(x - y)$$

Level curves:
For $z_0 \in [1, 3]$, we get a family of lines

Level curves for $z_0 = 2$:

$$y = x - k\,\pi, \quad k \in \mathbb{Z}$$

Level curves for $z_0 = 1$:

$$y = x - \frac{\pi}{2} + 2k\,\pi, \quad k \in \mathbb{Z}$$

Example 3 of level curves (3)

Function:

$$f(x, y) = 2 + \sin(x - y)$$

Depiction of level curves: