Functions of several variables

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Multivariate calculus - MA 261

Mostly taken from *Calculus, Early Transcendentals*
by Briggs - Cochran - Gillett - Schulz

Outline

Outline

Recalling functions of 1 variable (1)

Example of function:

$$y = f(x) = \sqrt{9 - x^2}$$

Questions:

1. Domain of f?
2. Range of f?

Recalling functions of 1 variable (2)

Recalling the function:

$$y = f(x) = \sqrt{9 - x^2}$$

Domain:

$$x \in [-3, 3]$$

Range:

$$y \in [0, 3]$$

Functions of 2 variables: example (1)

Example of function:

$$z = f(x, y) = \sqrt{9 - x^2} - \sqrt{25 - y^2}$$

Questions:

1. Domain of f?
2. Range of f?

Functions of 2 variables: example (2)

Recalling the function:

$$z = f(x, y) = \sqrt{9 - x^2} - \sqrt{25 - y^2}$$

Domain:

$$(x, y) \in [-3, 3] \times [-5, 5]$$

Range: Looking at lines $x = \pm 3$ and $y = \pm 5$, we get

$$y \in [-5, 3]$$

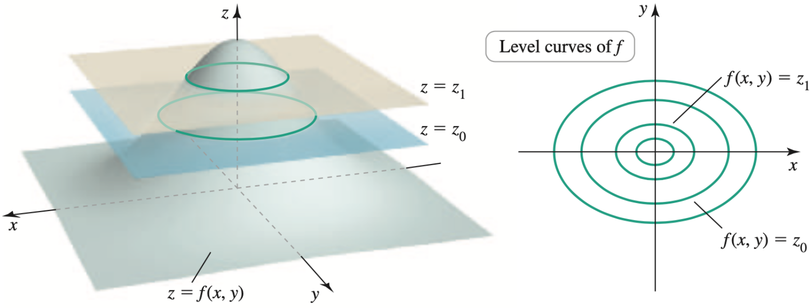
Contour and level curves

Definition 1.

Contour curve:
Intersection of the surface $(x, y, f(x, y))$ and plane $z = z_0$

Level curve:
Projection of contour curve on xy-plane

Contour and level curves: illustration



Example of level curves (1)

Function:

$$f(x,y) = y - x^2 - 1$$

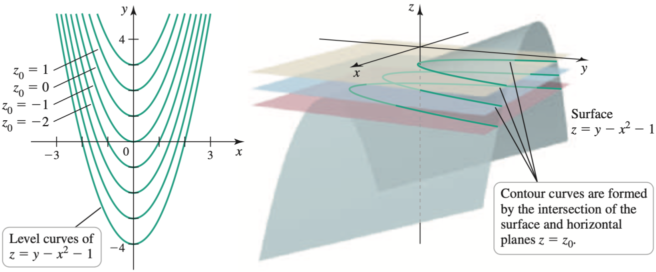
Example of level curves (2)

Function:

$$f(x,y) = y - x^2 - 1$$

Level curves: For $z_0 \in \mathbb{R}$, we get the parabola

$$y = x^2 + 1 + z_0$$



Example 2 of level curves (1)

Function:

$$f(x,y) = \exp\left(-x^2 - y^2\right)$$

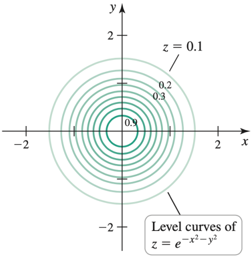
Example 2 of level curves (2)

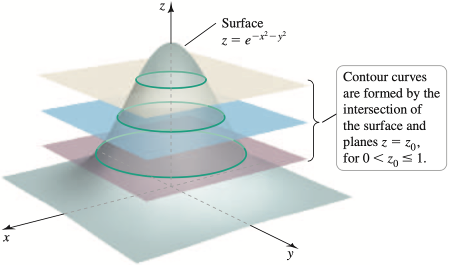
Function:

$$f(x,y) = \exp\left(-x^2 - y^2\right)$$

Level curves: For $z_0 \in (0,1]$, we get the circle

$$x^2 + y^2 = -\ln(z_0)$$

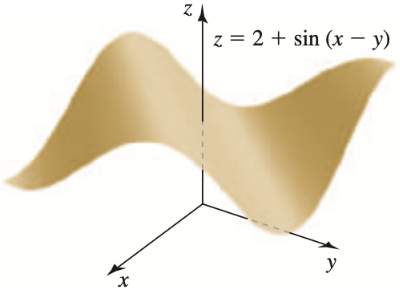




Example 3 of level curves (1)

Function:

$$f(x,y) = 2 + \sin(x - y)$$



Example 3 of level curves (2)

Function:

$$f(x,y) = 2 + \sin(x-y)$$

Level curves:
For $z_0 \in [1,3]$, we get a family of lines

Level curves for $z_0 = 2$:

$$y = x - k\,\pi, \quad k \in \mathbb{Z}$$

Level curves for $z_0 = 1$:

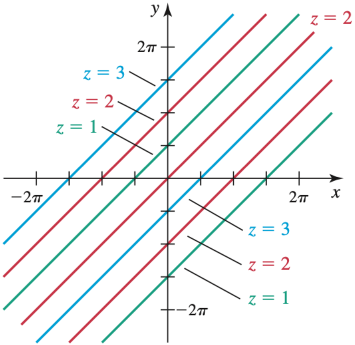
$$y = x - \frac{\pi}{2} + 2k\,\pi, \quad k \in \mathbb{Z}$$

Example 3 of level curves (3)

Function:

$$f(x, y) = 2 + \sin(x - y)$$

Depiction of level curves:



Application of functions of 2 variables (1)

Situation:

- Fraction of students infected by FV is r on $9/12$
- We have n random encounters with students on $9/12$

Function:

The probability of meeting at least one student with FV is

$$p(n, r) = 1 - (1 - r)^n$$

This requires probability theory and is admitted

Question:

Draw level curves

Application of functions of 2 variables (2)

Function:

$$p(n, r) = 1 - (1 - r)^n$$

Useful values of z:

For $p_0 \in [0, 1]$, the curve $p(n, r) = p_0$ is non empty

Level curves for $p_0 \in [0, 1]$:

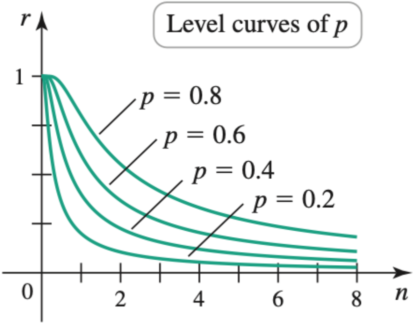
$$r = 1 - (1 - p)^{1/n}$$

Application of functions of 2 variables (3)

Function:

$$p(n, r) = 1 - (1 - r)^n$$

Depiction of level curves:

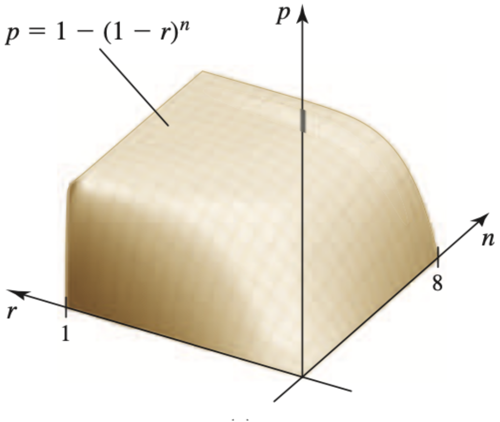


Application of functions of 2 variables (4)

Function:

$$p(n, r) = 1 - (1 - r)^n$$

Depiction of function:



Outline

Continuity for functions of 1 variable (1)

Limit: The assertion

$$\lim_{x \to a} f(x) = L$$

means that $f(x)$ can be made as close to L as we wish
$\hookrightarrow$ by making x close to a

Remark: If $\lim_{x \to a} f(x) = L$, then
the limit should not depend on the way $x \to a$

Continuity for functions of 1 variable (2)

Continuity: The function f is continuous at point a if

$$\lim_{x \to a} f(x) = f(a)$$

Examples of continuous functions:

- Polynomials
- sin, cos, exponential
- Rational functions (on their domain)
- Log functions (on their domain)

Continuity for functions of 2 variables (1)

Limit: The assertion

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

means that $f(x,y)$ can be made as close to L as we wish
$\hookrightarrow$ by making (x,y) close to (a,b)

Remark: If $\lim_{(x,y)\to(a,b)} f(x,y) = L$, then
the limit should not depend on the way $(x,y) \to (a,b)$

Continuity for functions of 2 variables (2)

Continuity: The function f is continuous at point a if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Examples of continuous functions:

- Polynomials
- sin, cos, exponential
- Rational functions (on their domain)
- Log functions (on their domain)

Logarithmic example (1)

Function:

$$\ln\left(\frac{1+y^2}{x^2}\right)$$

Problem: Continuity at point

$$(1,0)$$

Logarithmic example (2)

Continuity: f is the log of a rational function
$\hookrightarrow$ Continuous wherever it is defined

Definition at point $(1, 0)$: We have

$$f(1, 0) = 0$$

This is well defined

Conclusion: f is continuous at $(1, 0)$, that is

$$\lim_{(x,y)\to(1,0)} f(x, y) = f(1, 0) = 0$$

Rational function example (1)

Function:

$$f(x, y) = \frac{y^2 - 4x^2}{2x^2 + y^2}$$

Problem: Continuity at point

$$(0, 0)$$

Rational function example (2)

Continuity: f is a rational function
$\hookrightarrow$ Continuous wherever it is defined

Definition at point $(0,0)$: We have

$$f(0,0) = \frac{0}{0}$$

This is not well defined, therefore general result cannot be applied

Rational function example (3)

Two paths: We have

$$\begin{aligned} \text{Along } x = 0, \quad \lim_{(x,y)\to(0,0),\, x=0} \frac{y^2 - 4x^2}{2x^2 + y^2} &= 1 \\ \text{Along } y = 0, \quad \lim_{(x,y)\to(0,0),\, y=0} \frac{y^2 - 4x^2}{2x^2 + y^2} &= -2 \end{aligned}$$

We get 2 different limits

Conclusion:
f is not continuous at point $(0, 0)$

Another rational function example (1)

Function:

$$f(x, y) = \frac{x^2 - y^2}{x + y}$$

Problem: Continuity at point

$$(0, 0)$$

Another rational function example (2)

Continuity: f is a rational function
$\hookrightarrow$ Continuous wherever it is defined

Definition at point $(0, 0)$: We have

$$f(0, 0) = \frac{0}{0}$$

This is not well defined, therefore general result cannot be applied

Another rational function example (3)

Two paths: We have

$$\text{Along } x=0, \quad \lim_{(x,y)\to(0,0),\, x=0} \frac{x^2-y^2}{x+y} = 0$$

$$\text{Along } y=0, \quad \lim_{(x,y)\to(0,0),\, y=0} \frac{x^2-y^2}{x+y} = 0$$

We get the same limit

Partial conclusion:
This is not enough!

Another rational function example (4)

Next steps: Try different paths

- $y = x^2$, $y = x^3$, etc
- Those all give a 0 limit
- This is still not enough

Key remark: If $(x, y) \neq (0, 0)$ we have

$$f(x, y) = \frac{x^2 - y^2}{x + y} = x - y$$

The rhs above is continuous

Conclusion: We have

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0$$

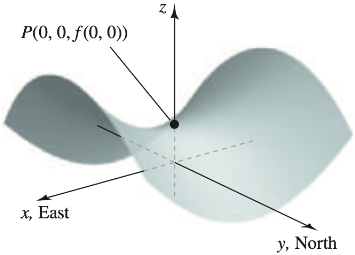
Outline

Motivation

Derivative for functions of 1 variable: Captures the rate of change

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Rate of change in the 2-d case: Can be different in x and y directions
$\hookrightarrow$ Captured by partial derivatives



Partial derivatives

Definition 2.

Consider

- f function of 2 variables

Then we set

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$
$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Some remarks on partial derivatives

Frozen and live variables:

- In order to compute $f_x(x, y)$
  $\hookrightarrow$ the x variable is alive and the y variable is frozen
- In order to compute $f_y(x, y)$
  $\hookrightarrow$ the y variable is alive and the x variable is frozen

Funny notation: For partial derivatives we also use

$$\frac{\partial f}{\partial x}(x, y) = f_x(x, y), \qquad \frac{\partial f}{\partial y}(x, y) = f_y(x, y)$$

Example of computation (1)

Function:

$$f(x, y) = x^8 y^5 + x^3 y$$

Example of computation (2)

Recall:

$$f(x, y) = x^8 y^5 + x^3 y$$

Partial derivative f_x:

$$f_x = 8x^7 y^5 + 3x^2 y$$

Partial derivative f_y:

$$f_y = 5x^8 y^4 + x^3$$

Second example of computation (1)

Function:

$$f(x, y) = e^x \sin(y)$$

Second example of computation (2)

Recall:

$$f(x, y) = e^x \sin(y)$$

Partial derivative f_x:

$$f_x = e^x \sin(y)$$

Partial derivative f_y:

$$f_y = e^x \cos(y)$$

Second derivatives

Second derivative f_{xx}, f_{yy}:

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2}, \qquad f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2}$$

Second derivative f_{xy}:

$$f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial x \partial y}$$

Second derivative f_{yx}:

$$f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial y \partial x}$$

Example of second derivatives

Function:

$$f(x, y) = e^x \sin(y)$$

Second derivative f_{xx}:

$$f_{xx} = (f_x)_x = e^x \sin(y)$$

Second derivative f_{xy}:

$$f_{xy} = (f_x)_y = e^x \cos(y)$$

Order of derivatives

On our running example: We have

$$f_{yx} = (f_y)_x = e^x \cos(y) = f_{xy}$$

General result (Clairaut's theorem):
For a smooth f, the order of the derivatives does not matter

$$f_{yx} = f_{xy}$$

Example of order of derivatives (1)

Function:

$$f(x, y) = e^{x^2 y}$$

Problem: Check that

$$f_{yx} = f_{xy}$$

Example of order of derivatives (2)

Recall:

$$f(x, y) = e^{x^2 y}$$

Partial derivative f_x:

$$f_x = 2xy\, e^{x^2 y}$$

Partial derivative f_y:

$$f_y = x^2\, e^{x^2 y}$$

Mixed derivatives:

$$f_{yx} = f_{xy} = 2x\left(x^2 y + 1\right) e^{x^2 y}$$

Functions of 3 variables (1)

Basic rule: Functions of 3 variables are handled
$\hookrightarrow$ in the same way as functions of 2 variables

Example:

$$f(x, y, z) = xyz$$

First derivatives:

$$f_x = yz, \qquad f_y = xz, \qquad f_z = xy$$

Functions of 3 variables (2)

Second derivatives: We have for instance

$$f_{xy} = f_{yx} = z$$

Third derivatives: The only non zero derivatives are

$$f_{xyz} = f_{xzy} = \cdots = f_{zyx} = 1$$

Outline

Chain rule for functions of 1 variable

Situation: We have

- $y = f(x)$
- $x = g(t)$

Chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}x}{\mathrm{d}t}$$

Chain rule with 1 independent variable

Theorem 3.

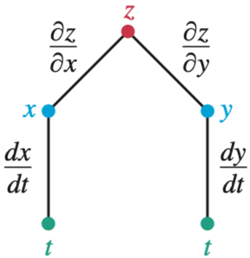
Let

- $z = z(x, y)$
- $x = x(t)$ and $y = y(t)$
- z, x, y differentiable

Then

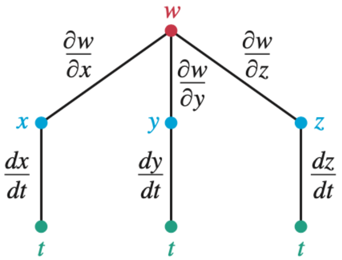
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Tree representation of chain rule (2d)



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Tree representation of chain rule (3d)



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example of computation (1)

Functions: We consider

$$z = x^2 - 3y^2 + 20, \quad x = 2\cos(t), \quad y = 2\sin(t)$$

Derivative: We find

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = -16\sin(2t)$$

Particular value: It $t = \frac{\pi}{4}$, then

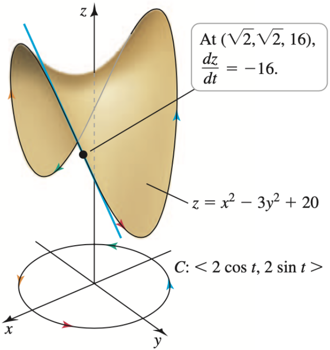
$$\frac{\mathrm{d}z}{\mathrm{d}t}\left(\frac{\pi}{4}\right) = -16$$

Example of computation (2)

Other possible strategy:

1. Express $z(x(t), y(t))$ as a function $F(t)$
2. Differentiate as usual

Problem: this becomes impractical very soon.



Implicit differentiation

Theorem 4.

Let $F(x, y)$ be such that

- F differentiable
- The equation $F(x, y) = 0$ defines $y = y(x)$
- $x \mapsto y(x)$ differentiable
- $F_y \neq 0$

Then we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$$

Example of implicit differentiation (1)

Equation:

$$e^y \sin(x) = x + xy$$

Problem: Find

$$\frac{\mathrm{d}y}{\mathrm{d}x}$$

Example of implicit differentiation (2)

Reformulation of the equation: $F(x, y) = 0$ with

$$F(x, y) = e^y \sin(x) - x - xy$$

Implicit differentiation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y} = -\frac{e^y \cos(x) - 1 - y}{e^y \sin(x) - x}$$

Implicit differentiation with 3 variables (1)

Implicit equation: We consider

- $F(x, y, z) = xy + yz + xz$
- Equation: $F(x, y) = 3$
- The equation defines $z = z(x, y)$

Problem: Find

$$\frac{\partial z}{\partial y}$$

Implicit differentiation with 3 variables (2)

Implicit differentiation:

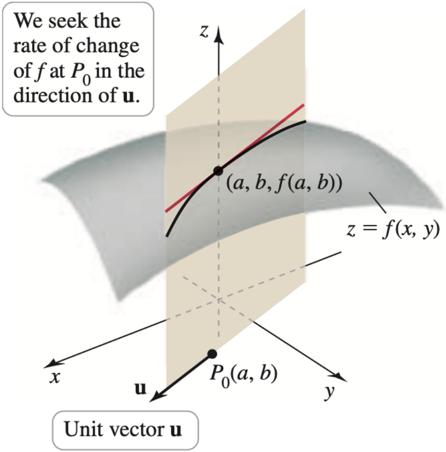
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x+z}{y+x}$$

Outline

Objective

Aim: Understand variations of a function

$\hookrightarrow$ In directions which are not parallel to the axes



Directional derivative

Definition 5.

Let

- f differentiable function at (a, b)
- $\mathbf{u} = \langle u_1, u_2 \rangle$ unit vector in xy-plane

Then the directional derivative of f in the direction of $\mathbf{u}$ at (a, b) is

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

Computation of the directional derivative

Proposition 6.

Let

- f differentiable function at (a, b)
- $\mathbf{u} = \langle u_1, u_2 \rangle$ unit vector in xy-plane

Then the directional derivative of f
in the direction of $\mathbf{u}$ at (a, b) is given by

$$D_{\mathbf{u}} f(a, b) = f_x(a, b) u_1 + f_y(a, b) u_2$$

Remark: One can also write

$$D_{\mathbf{u}} f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle$$

Example of directional derivative (1)

Function: Paraboloid of the form

$$z = f(x, y) = \frac{1}{4}\left(x^2 + 2y^2\right) + 2$$

Unit vector:

$$\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Problem: Compute the directional derivative

$$D_{\mathbf{u}} f(3, 2)$$

Example of directional derivative (2)

Function: Paraboloid of the form

$$z = f(x, y) = \frac{1}{4}\left(x^2 + 2y^2\right) + 2$$

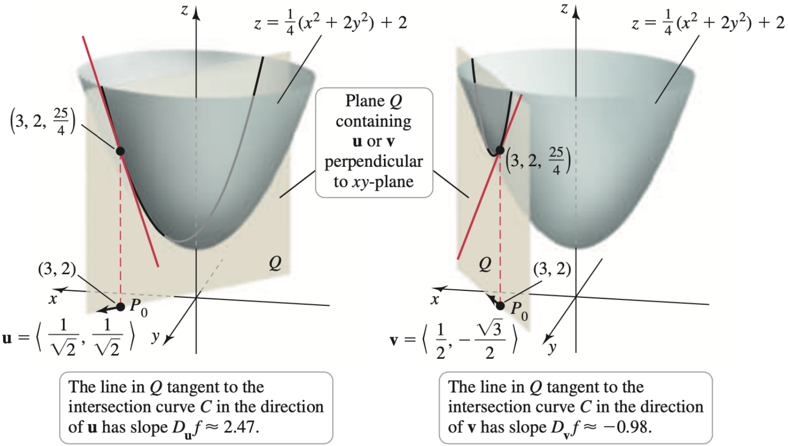
Unit vector:

$$\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Directional derivative: We get

$$D_{\mathbf{u}} f(3, 2) = \left\langle \frac{3}{2}, 2 \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{7}{2\sqrt{2}} \simeq 2.47$$

Example of directional derivative (3)



Gradient

Definition 7.

Let

- f differentiable function at (x, y)

Then the gradient of f at (x, y) is

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

Example of gradient (1)

Function:

$$f(x,y) = 3 - \frac{x^2}{10} + \frac{xy^2}{10}$$

Problem:

1. Compute $\nabla f(3,-1)$
2. Compute the directional derivative of f
   $\hookrightarrow$ at $(3,-1)$ in the direction of the vector $\langle 3,4 \rangle$

Example of gradient (2)

Gradient:

$$\nabla f(x,y) = \left\langle -\frac{x}{5} + \frac{y^2}{10}, \frac{xy}{5} \right\rangle$$

Thus

$$\nabla f(3,-1) = \left\langle -\frac{1}{2}, -\frac{3}{5} \right\rangle$$

Example of gradient (3)

Directional derivative: Unit vector in direction of $\langle 3, 4 \rangle$ is

$$\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

Thus directional derivative in direction of $\langle 3, 4 \rangle$ is

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

We get

$$D_{\mathbf{u}} f(3, -1) = -\frac{39}{50}$$

Interpretation of gradient

Remark: If

- $\mathbf{u}$ is a unit vector
- $\theta \equiv$ angle between $\mathbf{u}$ and $\nabla f(x, y)$

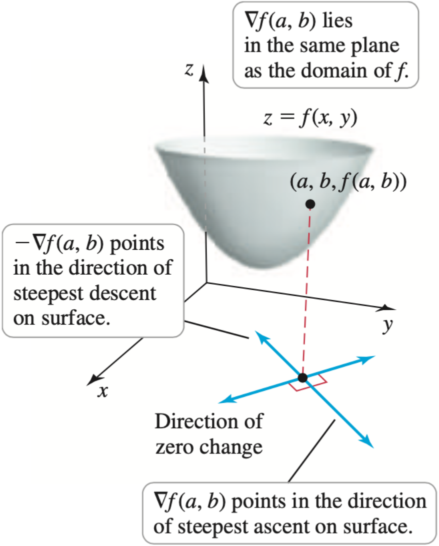
Then

$$D_{\mathbf{u}} f(x, y) = |\nabla f(x, y)| \cos(\theta)$$

Information given by the gradient

1. $|\nabla f(x, y)|$ is the maximal possible directional derivative
2. The direction $\mathbf{u} = \frac{\nabla f(x,y)}{|\nabla f(x,y)|}$ is the one of maximal ascent
3. The direction $\mathbf{u} = -\frac{\nabla f(x,y)}{|\nabla f(x,y)|}$ is the one of maximal desccent
4. If $\mathbf{u} \perp \nabla f(x, y)$, the directional derivative is 0

Interpretation of gradient: illustration



Example of steepest descent (1)

Function:

$$f(x, y) = 4 + x^2 + 3y^2$$

Questions:

1. If you are located on the paraboloid at the point $(2, -\frac{1}{2}, \frac{35}{4})$
   $\hookrightarrow$ In which direction should you move in order to ascend on the surface at the maximum rate?
2. If you are located on the paraboloid at the point $(2, -\frac{1}{2}, \frac{35}{4})$
   $\hookrightarrow$ In which direction should you move in order to descend on the surface at the maximum rate?
3. At the point (3, 1, 16), in what direction(s) is there no change in the function values?

Example of steepest descent (2)

Gradient:

$$\nabla f(x, y) = \langle 2x,\ 6y \rangle$$

Thus

$$\nabla f\left(2, -\frac{1}{2}\right) = \langle 4,\ -3 \rangle$$

Steepest ascent direction: We get

$$\mathbf{u} = \left\langle \frac{4}{5},\ -\frac{3}{5} \right\rangle,$$

with rate of ascent

$$\left| \nabla f\left(2, -\frac{1}{2}\right) \right| = 5$$

Example of steepest descent (3)

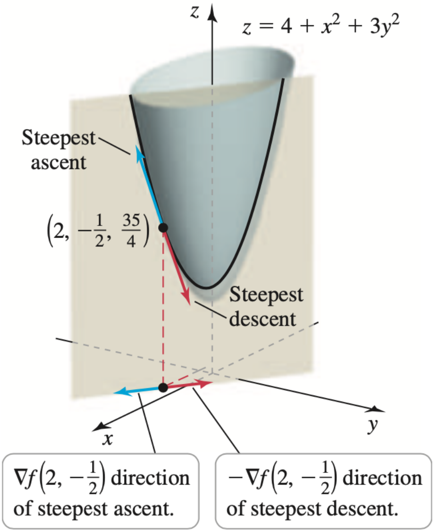
Steepest descent direction: We get

$$\mathbf{v} = -\mathbf{u} = \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle,$$

with rate of descent

$$-\left|\nabla f\left(2, -\frac{1}{2}\right)\right| = -5$$

Example of steepest descent (4)



Example of steepest descent (5)

Gradient at point $(3,1)$: Recall that

$$\nabla f(x,y) = \langle 2x,\, 6y \rangle$$

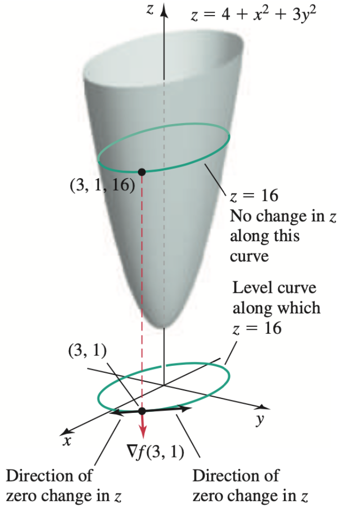
Thus

$$\nabla f\,(3,1) = \langle 6,\, 6 \rangle$$

Direction of 0 change: Any direction $\perp \langle 6,\, 6 \rangle$
$\hookrightarrow$ Unit vectors given by

$$\mathbf{u} = \frac{1}{\sqrt{2}} \langle 1,\, -1 \rangle\,, \qquad \frac{1}{\sqrt{2}} \langle -1,\, 1 \rangle$$

Example of steepest descent (6)



Gradient and level curves

Theorem 8.

Let

- f differentiable function at (x, y)
- Hypothesis: $\nabla f(a, b) \neq 0$

Then:

The line tangent to the level curve of f at (a, b)
is
orthogonal to $\nabla f(a, b)$

Hyperboloid example (1)

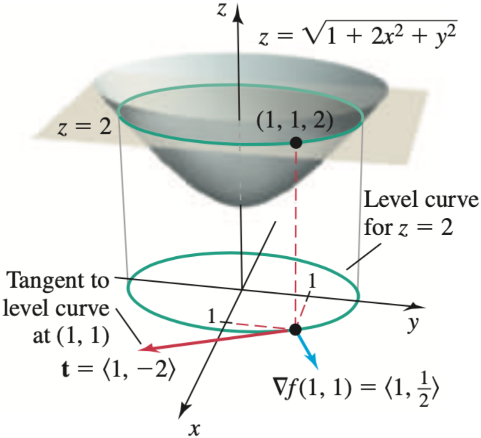
Function:

$$z = f(x, y) = \sqrt{1 + 2x^2 + y^2}$$

Questions:

1. Verify that the gradient at $(1, 1)$ is orthogonal to the corresponding level curve at that point.
2. Find an equation of the line tangent to the level curve at $(1, 1)$

Hyperboloid example (2)



$\mathbf{t} \cdot \nabla f = 0$

∇f is orthogonal to level curves.

Hyperboloid example (3)

Point on surface:
Given by $(1, 1, 2) \Longrightarrow$ On level curve $z = 2$

Equation for level curve: Ellipse of the form

$$1 + 2x^2 + y^2 = 4 \quad \Longleftrightarrow \quad 2x^2 + y^2 = 3$$

Implicit derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y} = -\frac{2x}{y}$$

Thus

$$\frac{\mathrm{d}y}{\mathrm{d}x}(1) = -2$$

Hyperboloid example (4)

Tangent vector: Proportional to

$$\mathbf{t} = \langle 1, -2 \rangle$$

Gradient of f:

$$\nabla f(x, y) = \left\langle \frac{2x}{\sqrt{1 + 2x^2 + y^2}}, \frac{y}{\sqrt{1 + 2x^2 + y^2}} \right\rangle$$

Thus

$$\nabla f(1, 1) = \left\langle 1, \frac{1}{2} \right\rangle$$

Orthogonality: We have

$$\mathbf{t} \cdot \nabla f(1, 1) = 0$$

Hyperboloid example (5)

Tangent line to level curve: At point $(1,1)$ we get

$$f_x(1,1)(x-1) + f_y(1,1)(y-1) = 0,$$

that is

$$y = -2x + 3$$

Generalization to 3 variables

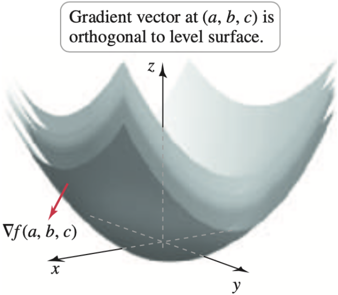
Situation:

- We have a function $w = f(x, y, z)$
- Each w_0 results in a level surface

$$f(x, y, z) = w_0$$

Gradient on level surface:

Will be $\perp$ to level surface



Example of tangent plane (1)

Function:

$$f(x, y, z) = xyz$$

Gradient:

$$\nabla f(x, y, z) = \langle yz, xz, xy \rangle$$

Thus

$$\nabla f(1, 2, 3) = \langle 6, 3, 2 \rangle$$

Example of tangent plane (2)

Plane tangent to level surface:

$$\langle 6, 3, 2 \rangle \cdot \langle x - 1, y - 2, z - 3 \rangle = 0$$

We get

$$6x + 3y + 2z = 18$$

Outline

Linear approximation for functions of 1 variable

Situation: We have

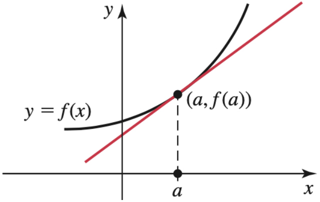
- $y = f(x)$

Tangent vector at a:

$$\mathbf{t} = (1, f'(a))$$

Linear approximation: Near a we have

$$f(x) \simeq f(a) + f'(a)(x - a)$$



Tangent plane for $F(x, y, z) = 0$

Definition 9.

Let $F(x, y, z)$ be such that

- F differentiable at $P(a, b, c)$
- $\nabla F \neq 0$
- S is the surface $F(x, y, z) = 0$

Then the tangent plane at (a, b, c) is given by

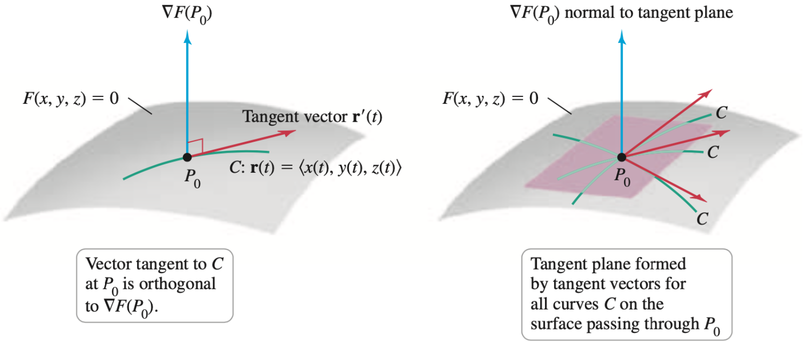
$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

Interpretation of tangent plane

Tangent plane as collection of tangent vectors: If

- S is the surface $F(x,y,z) = 0$
- $\mathbf{r}$ is a curve passing through (a,b,c) at time t

Then $\mathbf{r}'(t) \in$ tangent plane



Example of tangent plane (1)

Surface: Ellipsoid of the form

$$F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0$$

Questions:

1. Tangent plane at $(0, 4, \frac{3}{5})$
2. What tangent planes to S are horizontal?

Example of tangent plane (2)

Gradient: We have

$$\nabla F(x, y, z) = \left\langle \frac{2x}{9}, \frac{2y}{25}, 2z \right\rangle$$

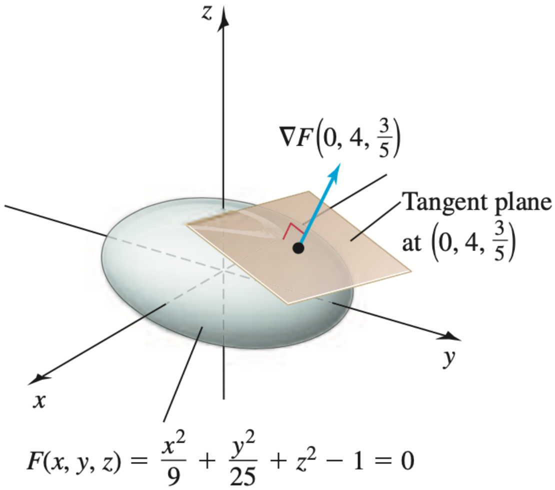
Thus

$$\nabla F(0, 4, \frac{3}{5}) = \left\langle 0, \frac{8}{25}, \frac{6}{5} \right\rangle$$

Tangent plane:

$$4y + 15z = 25$$

Example of tangent plane (3)



Example of tangent plane (4)

Horizontal plane: When the normal vector is of the form

$$\mathbf{n} = (0, 0, c), \quad \text{with} \quad c \neq 0$$

Horizontal tangent plane: When the normal vector ∇F is of the form

$$\nabla F(x, y, z) = (0, 0, c) \quad \Longleftrightarrow \quad F_x = 0,\ F_y = 0,\ F_z \neq 0$$

Solutions: Horizontal tangent plane for

$$(0, 0, 1) \quad \text{and} \quad (0, 0, -1)$$

Tangent plane for $z = f(x, y)$

Definition 10.

Let $f(x, y)$ be such that

- f differentiable at (a, b)
- S is the surface $z = f(x, y)$

Then the tangent plane to S at $(a, b, f(a, b))$ is given by

$$z = f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b) + f(a, b)$$

Example of tangent plane for $z = f(x, y)$ (1)

Surface: Paraboloid of the form

$$z = f(x, y) = 32 - 3x^2 - 4y^2$$

Question:

- Tangent plane at $(2, 1, 16)$

Example of tangent plane for $z = f(x, y)$ (2)

Partial derivatives: We have

$$f_x = 6x, \qquad f_y = -8y$$

Thus

$$f_x(2,1) = -12, \qquad f_y(2,1) = -8$$

Tangent plane:

$$z = -12x - 8y + 48$$

Linear approx for functions of 1 variable (Repeat)

Situation: We have

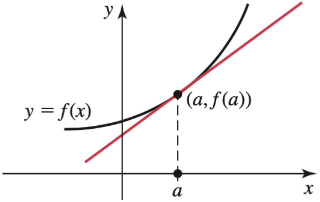
- $y = f(x)$

Tangent vector at a:

$$\mathbf{t} = (1, f'(a))$$

Linear approximation: Near a we have

$$f(x) \simeq f(a) + f'(a)(x - a)$$



Linear approximation for functions of 2 variables

Definition 11.

Let $f(x, y)$ be such that

- f differentiable at (a, b)
- S is the surface $z = f(x, y)$

Then the linear approximation to S at $(a, b, f(a, b))$ is given by

$$L(x, y) = f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b) + f(a, b)$$

Remark: Another popular form of the linear approximation is

$$\Delta z \simeq f_x \mathrm{d}x + f_y \mathrm{d}y$$

Example of infinitesimal change (1)

Function:

$$z = f(x, y) = x^2 y$$

Question: Evaluate the percentage of change in z if

- x is increased by 1%
- y is decreased by 3%

Example of infinitesimal change (2)

Small change in z:

$$\mathrm{d}z \simeq f_x \mathrm{d}x + f_y \mathrm{d}y = 2xy\mathrm{d}x + x^2\mathrm{d}y$$

Small percentage change in z:

$$\frac{\mathrm{d}z}{z} = \frac{2xy}{z}\,\mathrm{d}x + \frac{x^2}{z}\,\mathrm{d}y = \frac{2}{x}\,\mathrm{d}x + \frac{1}{y}\,\mathrm{d}y$$

If $\frac{\mathrm{d}x}{x} = .01$ and $\frac{\mathrm{d}y}{y} = -.03$:

$$\frac{\mathrm{d}z}{z} = -.01 = -1\%$$

Outline

Max and min for functions of 1 variable

Situation: We have

- $y = f(x)$

Critical point: $(c, f(c))$ whenever

$$f'(c) = 0$$

Second derivative test: If $(c, f(c))$ is critical then

1. If $f''(c) > 0$, there is a local minimum
2. If $f''(c) < 0$, there is a local maximum
3. If $f''(c) = 0$, the test is inconclusive

Critical points for functions of 2 variables

Definition 12.

Let

- f function of 2 variables
- (a, b) interior point in the domain of f

Then (a, b) is a critical point of f if

$$f_x(a, b) = 0, \quad \text{and} \quad f_y(a, b) = 0,$$

or if one of the partial derivatives f_x, f_y does not exist at (a, b)

Second derivative test

Theorem 13.

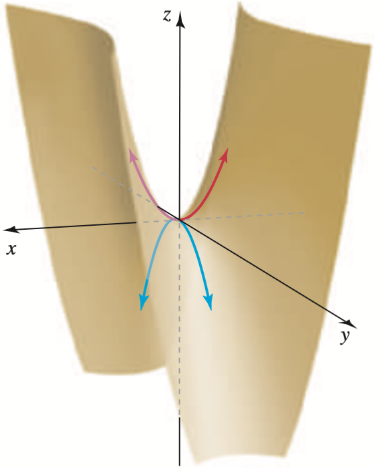
For f twice diff. function, define the discriminant of f as

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2$$

Then for a critical point (a,b) the following holds true:

1. If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, we have a local max
2. If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, we have a local min
3. If $D(a,b) < 0$, we have a saddle point
4. If $D(a,b) = 0$, the test is inconclusive

Saddle point for an hyperboloid



The hyperbolic paraboloid $z = x^2 - y^2$ has a saddle point at (0, 0).

Hyperboloids in architecture

Hyperboloids in the food industry

Example of critical points analysis (1)

Function:

$$f(x, y) = x^2 + 2y^2 - 4x + 4y + 6$$

Problem:

Use second derivative test to classify the critical points of f

Example of critical points analysis (2)

Partial derivatives:

$$f_x = 2x - 4, \qquad f_y = 4y + 4$$

Critical point:

$$(2, -1)$$

Critical value of f:

$$f(2, -1) = 0$$

Example of critical points analysis (3)

Second derivatives:

$$f_{xx} = 2, \qquad f_{xy} = f_{yx} = 0, \qquad f_{yy} = 4$$

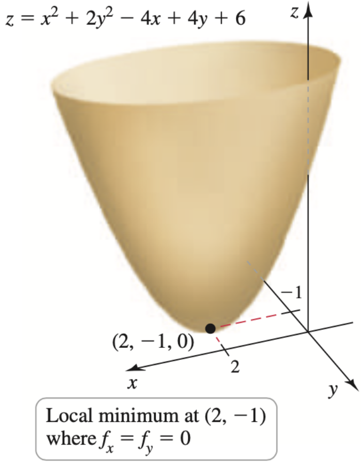
Discriminant:

$$D(x, y) = 8 > 0$$

Second derivative test: We have

$$D(2, -1) > 0, \; f_{xx}(2, -1) > 0 \quad \Longrightarrow \quad \text{Local minimum at } (2, -1)$$

Example of critical points analysis (4)



Second example (1)

Function:

$$f(x,y) = xy(x-2)(y+3)$$

Problem:

Use second derivative test to classify the critical points of f

Second example (2)

Partial derivatives:

$$f_x = 2y(x-1)(y+3), \qquad f_y = x(x-2)(2y+3)$$

Critical points:

$$(0,0), \quad (2,0), \quad (1,-\frac{3}{2}), \quad (0,-3), \quad (2,-3),$$

Second example (3)

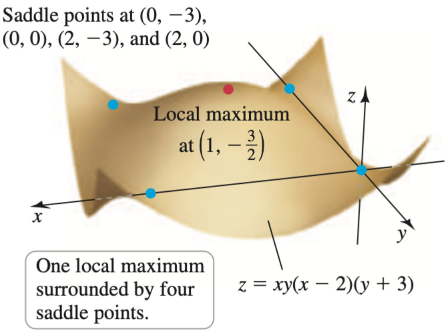
Second derivatives:

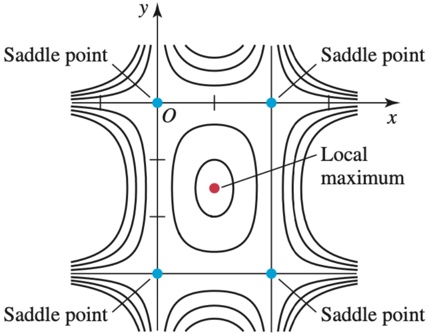
$$f_{xx} = 2y(y+3), \quad f_{xy} = 2(2y+3)(x-1), \qquad f_{yy} = 2x(x-2)$$

Analysis of critical points:

(x, y)	$D(x, y)$	f_{xx}	**Conclusion**
$(0, 0)$	-36	0	Saddle point
$(2, 0)$	-36	0	Saddle point
$(1, -3/2)$	9	$-9/2$	Local maximum
$(0, -3)$	-36	0	Saddle point
$(2, -3)$	-36	0	Saddle point

Second example (4)





Absolute maximum

Proposition 14.

Let

- f continuous function of 2 variables
- R closed region of $\mathbb{R}^2$

In order to find the maximum of f in R, we proceed as follows:

1. Determine the values of f at all critical points in R.
2. Find the maximum and minimum values of f on the boundary of R.
3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on R.

Example of global maximum (1)

Function:

$$z = f(x, y) = x^2 + y^2 - 2x - 4y$$

Region:

$$R = \{(x, y);\ (x, y) \text{ within triangle with vertices } (0,0), (0,4), (4,0)\}$$

Question:
Find global maximum of f on region R

Example of global maximum (2)

Partial derivatives:

$$f_x = 2x - 2, \qquad f_y = 2y - 4$$

Critical point:

$$(1, 2), \quad \text{with} \quad f(1, 2) = -5$$

Example of global maximum (3)

Boundary 1: On $y = 0$, $0 \leq x \leq 4$ we have

$$f(x,y) = x^2 - 2x \equiv g(x), \qquad g'(x) = 2(x-1)$$

Points of interest on boundary 1: We get

$$(0,0), \qquad (1,0), \qquad (0,4)$$

and

$$f(0,0) = 0, \qquad f(1,0) = -1, \qquad f(4,0) = 8$$

Example of global maximum (4)

Boundary 2: On $y = 4 - x$, $0 \leq x \leq 4$ we have

$$f(x, y) = 2x^2 - 6x \equiv h(x), \qquad h'(x) = 4x - 6$$

Points of interest on boundary 2: We get

$$(0, 4), \qquad \left(\frac{3}{2}, \frac{5}{2}\right), \qquad (4, 0)$$

and

$$f(0, 4) = 0, \qquad f\left(\frac{3}{2}, \frac{5}{2}\right) = -\frac{9}{2}, \qquad f(4, 0) = 8$$

Example of global maximum (5)

Boundary 3: On $x = 0$, $0 \leq y \leq 4$ we have

$$f(x, y) = y^2 - 4y \equiv k(y), \qquad k'(y) = 2(y - 2)$$

Points of interest on boundary 3: We get

$$(0, 0), \qquad (0, 2), \qquad (0, 4)$$

and

$$f(0, 0) = 0, \qquad f(0, 2) = -4, \qquad f(0, 4) = 0$$

Example of global maximum (6)

Summary of points of interest:

$$f(0,0)=0, \qquad f(1,0)=-1, \qquad f(4,0)=8$$
$$f(0,4)=0, \qquad f\left(\frac{3}{2},\frac{5}{2}\right)=-\frac{9}{2}, \qquad f(4,0)=8$$
$$f(0,0)=0, \qquad f(0,2)=-4, \qquad f(0,4)=0, \qquad f(1,2)=-5$$

Absolute minimum: at $(1,2)$ and

$$f(1,2)=-5$$

Absolute maximum: at $(4,0)$ and

$$f(4,0)=8$$

Outline

Global aim

Objective function:

$$f = f(x, y)$$

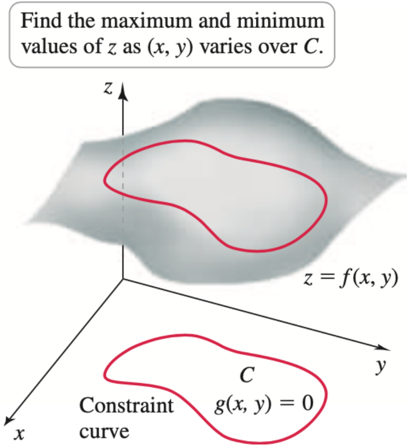
Constraint: We are moving on a curve of the form

$$g(x, y) = 0$$

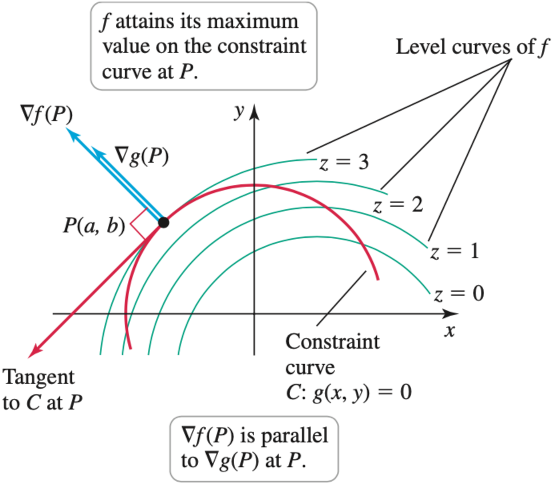
Optimization problem: Find

$$\max f(x, y), \quad \text{subject to} \quad g(x, y) = 0$$

Optimization problem: illustration



Lagrange multipliers intuition (1)



Lagrange multipliers intuition (2)

Some observations from the picture:

1. $P(a, b)$ on the level curve of f
   $\Longrightarrow$ Tangent to level curve $\perp \nabla f(a, b)$
2. $P(a, b)$ gives a maximum of f on curve C
   $\Longrightarrow$ Tangent to level curve $\parallel$ Tangent to constraint curve
3. Constraint is $g(x, y) = 0$
   $\Longrightarrow$ Tangent to constraint curve $\perp \nabla g(a, b)$

Conclusion (Lagrange's idea):
At the maximum under constraint we have

$$\nabla f(a, b) \parallel \nabla g(a, b)$$

Lagrange multipliers procedure

Optimization problem: Find

$$\max f(x, y), \quad \text{subject to} \quad g(x, y) = 0$$

Recipe:

1. Find the values of x, y and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y), \quad \text{and} \quad g(x, y) = 0$$

2. Select the largest and smallest corresponding function values.
$\hookrightarrow$ We get absolute max and min values of f s.t constraint.

Example of Lagrange multipliers (1)

Optimization problem: Find

$$\max f(x, y), \quad \text{with} \quad f(x, y) = x^2 + y^2 + 2,$$

subject to the constraint

$$g(x, y) = x^2 + xy + y^2 - 4 = 0$$

Example of Lagrange multipliers (2)

Computing the gradients: We get

$$\nabla f(x, y) = \langle 2x, 2y \rangle, \qquad \nabla g(x, y) = \langle 2x + y, x + 2y \rangle$$

Lagrange constraint 1:

$$f_x = \lambda g_x \iff 2x = \lambda (2x + y) \tag{1}$$

Lagrange constraint 2:

$$f_y = \lambda g_y \iff 2y = \lambda (x + 2y) \tag{2}$$

Example of Lagrange multipliers (3)

System for x, y: Gathering (1) and (2), we get

$$2(\lambda - 1)x + \lambda y = 0, \qquad \lambda x + 2(\lambda - 1)y = 0$$

This has solution $(0, 0)$ unless

$$\lambda = 2, \quad \text{or} \quad \lambda = \frac{2}{3}$$

Example of Lagrange multipliers (4)

Case $\lambda = 2$: We get $x = -y$. The constraint

$$x^2 + xy + y^2 - 4 = 0$$

becomes

$$x^2 - 4 = 0$$

Solutions:

$$x = 2, \quad \text{and} \quad x = -2$$

Corresponding values of f: We have

$$f(2, -2) = f(-2, 2) = 10$$

Example of Lagrange multipliers (5)

Case $\lambda = \frac{2}{3}$: We get $x = y$. The constraint

$$x^2 + xy + y^2 - 4 = 0$$

becomes

$$3x^2 - 4 = 0$$

Solutions:

$$x = \frac{2}{\sqrt{3}}, \quad \text{and} \quad x = -\frac{2}{\sqrt{3}}$$

Corresponding values of f: We have

$$f\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = f\left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right) = \frac{14}{3}$$

Example of Lagrange multipliers (6)

Absolute maximum:

For function f on the curve C defined by $g = 0$,

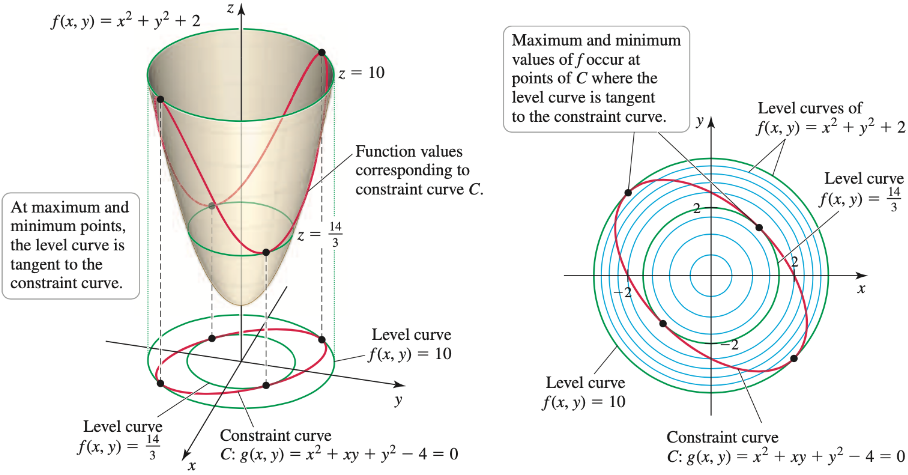
$$\text{Maximum} = 10, \quad \text{obtained for} \quad (2, -2), \ (-2, 2)$$

Absolute minimum:

For function f on the curve C defined by $g = 0$,

$$\text{Minimum} = \frac{14}{3}, \quad \text{obtained for} \quad \left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right), \ \left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$$

Example of Lagrange multipliers (7)



Optimization in dimension 3 (1)

Problem: Find the point on the sphere

$$x^2 + y^2 + z^2 = 1,$$

closest to the point

$$(1, 2, 3)$$

Optimization in dimension 3 (2)

Related minimization problem:

Find

$$\min f(x, y), \quad \text{with} \quad f(x, y) = (x-1)^2 + (y-2)^2 + (z-3)^2,$$

subject to the constraint

$$g(x, y) = x^2 + y^2 + z^2 - 1 = 0$$

Optimization in dimension 3 (3)

Computing the gradients: We get

$$\begin{aligned}\nabla f(x,y) &= \langle 2(x-1), 2(y-2), 2(z-3)\rangle \\ \nabla g(x,y) &= \langle 2x, 2y, 2z\rangle\end{aligned}$$

Lagrange constraint: We have

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$
$$\Longleftrightarrow$$
$$(\lambda-1)x = -1, \quad (\lambda-1)y = -2, \quad (\lambda-1)z = -3$$

Optimization in dimension 3 (4)

Solutions of Lagrange constraints:
The Lagrange system has unique solution whenever $\lambda \neq 1$. We get

$$x = -\frac{1}{\lambda - 1}, \quad y = -\frac{2}{\lambda - 1} = 2x, \quad z = -\frac{1}{\lambda - 1} = 3x$$

Reporting in constraint g: We have

$$y = 2x, \quad z = 3x, \quad g(x, y) = 0,$$

Thus we get

$$14x^2 = 1$$

Optimization in dimension 3 (5)

Solutions:

$$x = \frac{1}{\sqrt{14}}, \quad \text{and} \quad x = -\frac{1}{\sqrt{14}}$$

Corresponding values of f: We have

$$\begin{aligned} f\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) &\simeq 7.51 \\ f\left(-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right) &\simeq 22.48 \end{aligned}$$

Optimization in dimension 3 (6)

Absolute maximum:

Maximal distance from $(1,2,3)$ to a point on the sphere is

$$\text{Maximum} = 4.74, \quad \text{obtained for} \quad \left(-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right)$$

Absolute minimum:

Minimal distance from $(1,2,3)$ to a point on the sphere is

$$\text{Minimum} = 2.74 = \sqrt{7.51}, \quad \text{obtained for} \quad \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$$