Multiple integration

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Multivariate calculus - MA 261

Mostly taken from *Calculus, Early Transcendentals* by Briggs - Cochran - Gillett - Schulz

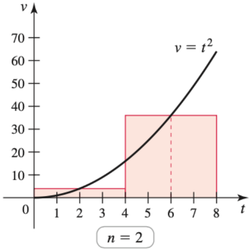
Outline

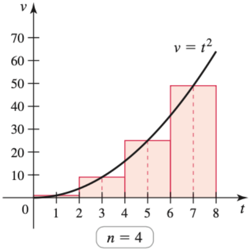
Outline

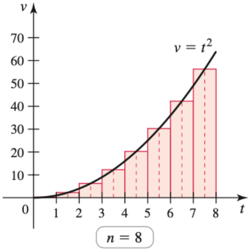
Integration in dimension 1 (1)

Approximation procedure:

Area under a curve is approximated by sum of rectangle areas



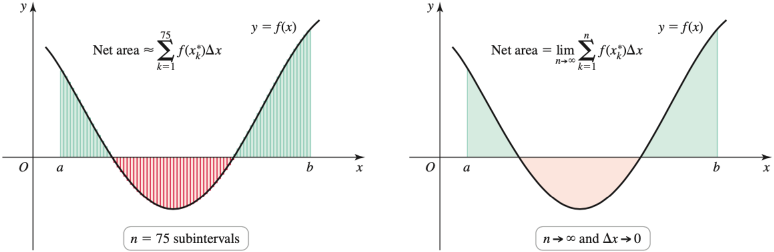




Integration in dimension 1 (2)

Riemann integral: In the limit we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x_k = \int_a^b f(x)\,\mathrm{d}x$$



Volume approximation (1)

Aim: Approximate the volume V
$\hookrightarrow$ Under the surface defined by f on rectangle $R = [a, b] \times [c, d]$

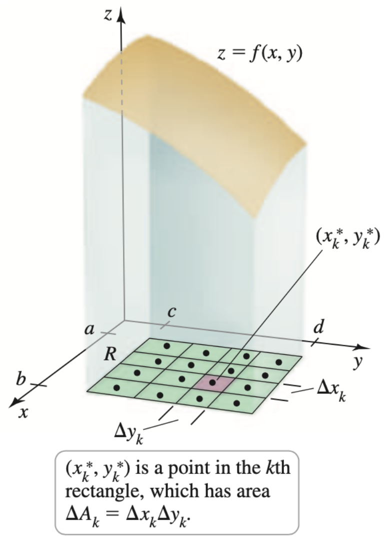
Approximation:

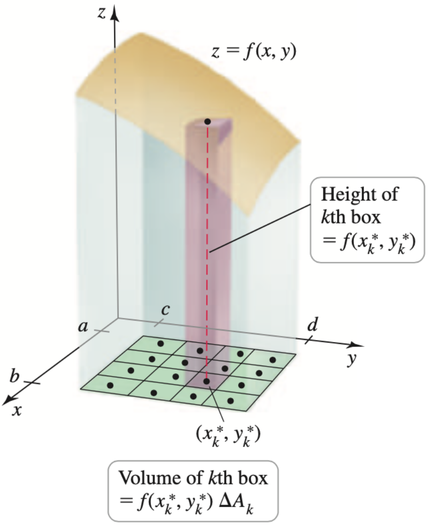
- Divide R into boxes centered at (x_k^*, y_k^*)
- Area of each box: $\Delta A_k = \Delta x_k \Delta y_k$

Then the volume is approximated as

$$V \simeq \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k$$

Volume approximation (2)

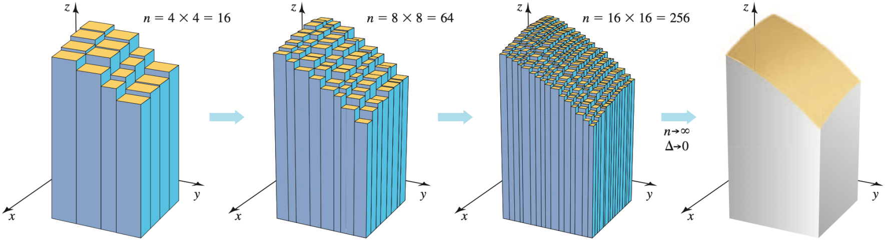




Integration in dimension 1 (3)

Double integral: In the limit we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k = \int\int_R f(x,y)\, \mathrm{d}A$$



Computing double integrals

Basic recipe:

1. Integrate inside out
2. While integrating wrt one variable, keep the other one constant
3. Fubini: The order of integration does not matter

Example of double integration (1)

Function:

$$z = f(x,y) = 6 - 2x - y$$

Region: Rectangle

$$R = [0,1] \times [0,2]$$

Problem: Compute

$$\int\int_R f(x,y)\,\mathrm{d}A$$

Example of double integration (2)

Integrating: We get

$$\int\int_R f(x,y)\,\mathrm{d}A = \int_0^1 \left(\int_0^2 (6-2x-y)\ \mathrm{d}y\right)\mathrm{d}x$$
$$= \int_0^1 (10-4x)\,\mathrm{d}x$$
$$= 10x - 2x^2\Big|_0^1$$

Area: We get

$$\int\int_R f(x,y)\,\mathrm{d}A = 8$$

To be checked: We also have

$$\int\int_R f(x,y)\,\mathrm{d}A = \int_0^2 \left(\int_0^1 (6-2x-y)\ \mathrm{d}x\right)\mathrm{d}y$$

Illustration: integrating first in y

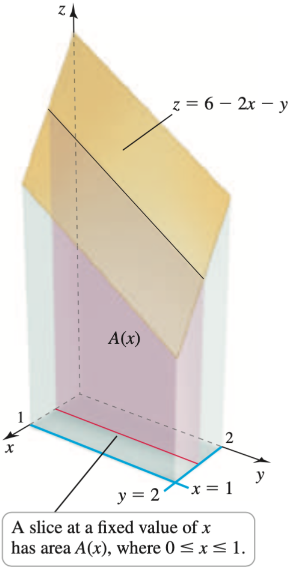
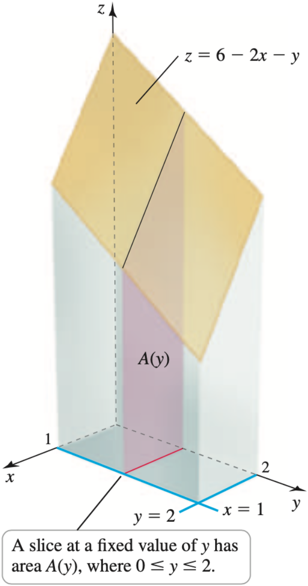


Illustration: integrating first in x



Choosing the correct order of integration (1)

Function:

$$z = f(x, y) = y^5 x^2 e^{x^3 y^3}$$

Region: Rectangle

$$R = [0, 2] \times [0, 1]$$

Problem: Compute

$$\int\int_R f(x, y) \, \mathrm{d}A$$

Choosing the correct order of integration (2)

Order of integration: We integrate wrt x first and compute

$$\begin{aligned}
\int\int_R f(x,y)\,\mathrm{d}A &= \int_0^1 y^2 \left(\int_0^2 y^3 x^2 e^{x^3 y^3}\,\mathrm{d}x\right)\mathrm{d}y \\
&= \frac{1}{3}\int_0^1 y^2 \left(e^{x^3 y^3}\Big|_{x=0}^{x=2}\right)\mathrm{d}y \\
&= \frac{1}{3}\int_0^1 y^2 \left(e^{8y^3} - 1\right)\mathrm{d}y \\
&= \frac{1}{72}e^8 - \frac{1}{9} \\
&\simeq 41.29
\end{aligned}$$

Average value

Definition 1.

Let

- f function of 2 variables
- R rectangle

Then the average value of f on R is given by

$$\bar{f} = \frac{1}{\text{Area}(R)} \int\int_R f(x, y) \, \mathrm{d}A$$

Example of average value (1)

Function:

$$z = f(x, y) = 2 - x - y$$

Region: Rectangle

$$R = [0, 2] \times [0, 2]$$

Problem:
Compute the average value of f on R

Example of average value (2)

Integrating: We get

$$\begin{aligned}\bar{f} &= \frac{1}{\text{Area}(R)} \int\int_R f(x,y)\,\mathrm{d}A \\ &= \frac{1}{4}\int_0^2 \left(\int_0^2 (2-x-y)\,\mathrm{d}x\right)\mathrm{d}y \\ &= \frac{1}{4}\int_0^2 (2-2y)\,\mathrm{d}y \\ &= 0\end{aligned}$$

Average value: We find that f is centered on R, ie

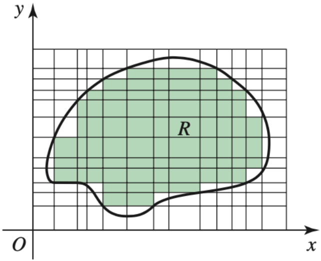
$$\bar{f} = 0$$

Outline

Description of the problem

New situation:
The region R of integration is not a rectangle

Consequence: Order of integration is important
$\hookrightarrow$ and cannot be switched arbitrarily



Special form of domain

Particular case: We have

$$R = \left\{ (x, y) \in \mathbb{R}^2;\ x \in [a, b],\ f(x) \leq y \leq g(x) \right\}$$

Recipe:

Integrate wrt variable with constant bounds last

Example of integration (1)

Function:

$$z = f(x, y) = xy^2$$

Region: Of the form

$$R = \left\{ (x, y) \in \mathbb{R}^2;\ x \in [0, \sqrt{2}],\ \ x^2 \le y \le 2 \right\}$$

Problem: Compute

$$\int\int_R f(x, y)\, \mathrm{d}A$$

Example of integration (2)

Order of integration: We integrate wrt y first and compute

$$\begin{aligned}
\int\int_R f(x,y)\,\mathrm{d}A &= \int_0^{\sqrt{2}} \left(\int_{x^2}^{2} xy^2\,\mathrm{d}y \right) \mathrm{d}x \\
&= \int_0^{\sqrt{2}} \frac{8}{3}x - \frac{1}{3}x^7\,\mathrm{d}x \\
&= \frac{8}{6}x^2 - \frac{1}{24}x^8 \Big|_0^{\sqrt{2}} \\
&= 2
\end{aligned}$$

Example of integration (3)

Switching order of integration:

One has to be more careful than for rectangles. We get that

$$R = \left\{(x, y) \in \mathbb{R}^2;\ x \in [0, \sqrt{2}],\ x^2 \le y \le 2\right\}$$

can also be written as

$$R = \left\{(x, y) \in \mathbb{R}^2;\ y \in [0, 2],\ 0 \le x \le \sqrt{y}\right\}$$

Example of integration (4)

Integration with order switched: We integrate wrt x first and compute

$$\int\int_R f(x,y)\,\mathrm{d}A = \int_0^2 \left(\int_0^{\sqrt{y}} xy^2\,\mathrm{d}x \right) \mathrm{d}y = \frac{1}{2}\int_0^2 y^3\,\mathrm{d}y = \frac{1}{8}\, y^4 \Big|_0^2 = 2$$

Switching order of integration (1)

Function: consider a general function

$$z = f(x, y)$$

Region: Of the form

$$R = \left\{(x, y) \in \mathbb{R}^2;\ x \in [0, 2],\ e^{-x} \leq y \leq e^{x}\right\}$$

Problem: Switch the order of integration for

$$\int\int_R f(x, y)\,\mathrm{d}A = \int_0^2 \int_{e^{-x}}^{e^{x}} f(x, y)\,\mathrm{d}y\mathrm{d}x$$

Switching order of integration (2)

Changing the definition of R: We have

$$\begin{aligned} R &= \left\{(x,y) \in \mathbb{R}^2;\ x \in [0,2],\ e^{-x} \leq y \leq e^x\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2;\ y \in [e^{-2},1],\ -\ln(y) \leq x \leq 2\right\} \\ &\quad \bigcup \left\{(x,y) \in \mathbb{R}^2;\ y \in [1,e^2],\ \ln(y) \leq x \leq 2\right\} \end{aligned}$$

New formula for the integral:

$$\int\int_R f(x,y)\,\mathrm{d}A = \int_{e^{-2}}^{1}\int_{-\ln(y)}^{2} f(x,y)\,\mathrm{d}x\mathrm{d}y + \int_{1}^{e^2}\int_{\ln(y)}^{2} f(x,y)\,\mathrm{d}x\mathrm{d}y$$

Choosing order of integration (1)

Function:

$$z = f(x, y) = \sin\left(x^2\right)$$

Region: Of the form

$$R = \left\{(x, y) \in \mathbb{R}^2;\ y \in [0, \sqrt{\pi}],\ \ y \le x \le \sqrt{\pi}\right\}$$

Problem: Compute

$$\int\int_R f(x, y)\, \mathrm{d}A$$

Choosing order of integration (2)

Impossible computation: Write

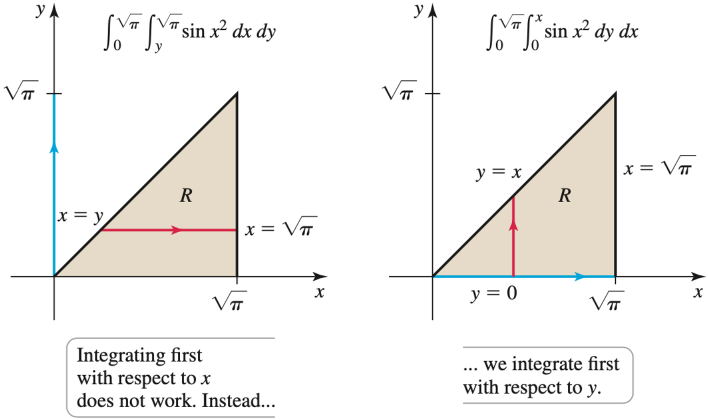
$$\int\int_R f(x,y)\,\mathrm{d}A = \int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}} \sin\left(x^2\right)\,\mathrm{d}x\mathrm{d}y$$

Then antiderivative of $\sin(x^2)$ not known!

Solution: Switch order of integration, ie write

$$R = \left\{(x,y)\in\mathbb{R}^2;\ x\in[0,\sqrt{\pi}],\ 0\le y\le x\right\}$$

Choosing order of integration (3)



Choosing order of integration (3)

Computing the integral:

$$\begin{aligned}\int\int_R f(x,y)\,\mathrm{d}A &= \int_0^{\sqrt{\pi}}\int_0^x \sin\left(x^2\right)\,\mathrm{d}y\mathrm{d}x\\ &= \int_0^{\sqrt{\pi}} x\sin\left(x^2\right)\,\mathrm{d}x\\ &= -\frac{1}{2}\cos\left(x^2\right)\Big|_0^{\sqrt{\pi}}\\ &= 1\end{aligned}$$

Remark: This trick does not always work!

Outline

Recalling polar coordinates

Cartesian coordinates: (x, y)

Polar coordinates: (r, θ) with

- $r \equiv$ distance from origin
- $\theta \equiv$ angle wrt x-axis

Polar to Cartesian: We have

$$x = r \cos(\theta), \qquad y = r \sin(\theta).$$

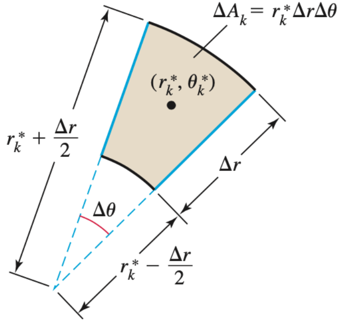
Cartesian to polar: We have

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Area of a small pizza crust (1)

Recall: For integration in Cartesian coordinates
$\hookrightarrow$ We used area of small rectangles $\Delta x \Delta y$

New aim: Find area of a small rectangle in polar coordinates



Area of a small pizza crust (2)

Approximation: If Δr and $\Delta\theta$ are small, then

$$\begin{aligned}\text{Area(Pizza crust)} &\simeq \text{Area(Small rectangle)} \\ &= \Delta r \, (r \, \Delta\theta) \\ &= r \, \Delta r \, \Delta\theta\end{aligned}$$

Polar change of coordinates

Theorem 2.

Let

- $f(x, y)$ continuous function
- R polar region of the form

$$R = \{(r, \theta);\ a \le r \le b,\ \alpha \le \theta \le \beta\}$$

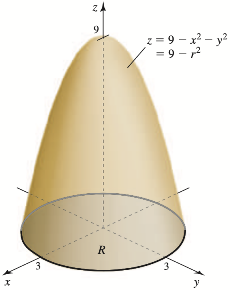
Then we have

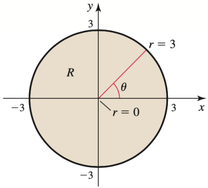
$$\int\int_R f(x, y)\,\mathrm{d}A = \int_\alpha^\beta \int_a^b f\left(r\cos(\theta), r\sin(\theta)\right) r\,\mathrm{d}r\mathrm{d}\theta$$

Computing a volume (1)

Problem: Find the volume bounded by

- Paraboloid $z = 9 - x^2 - y^2$
- xy-plane





Computing a volume (2)

Intersection with xy-plane: Circle defined by

$$x^2 + y^2 = 9$$

Polar coordinates domain:

$$R = \{(r, \theta);\ 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$$

Computing a volume (3)

Volume as an integral: We have

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^3 \left(9 - r^2\right) \mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{9}{2} r^2 - \frac{1}{4} r^4 \Big|_0^3 \mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{81}{4} \mathrm{d}\theta \end{aligned}$$

Thus

$$V = \frac{81\pi}{2}$$

Example of polar integral (1)

Problem: Compute

$$I = \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \left(x^2 + y^2\right)^{3/2} \mathrm{d}x\mathrm{d}y$$

Remark:
The integral looks terrible in Cartesian coordinates!

Example of polar integral (2)

Domain in Cartesian coordinates:

$$R = \left\{(x, y);\ -1 \leq y \leq 1,\ 0 \leq x \leq \sqrt{1 - y^2}\right\}$$

Domain in polar coordinates:

$$R = \left\{(r, \theta);\ 0 \leq r \leq 1,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right\}$$

Example of polar integral (3)

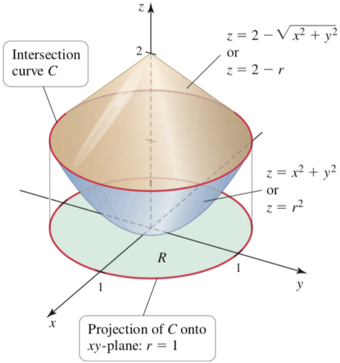
Integral in polar coordinates: We get

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r^3 \, r \, \mathrm{d}r \mathrm{d}\theta = \frac{\pi}{5}$$

Region bounded by 2 surfaces (1)

Problem: Find the volume bounded by

- Paraboloid $z = x^2 + y^2$
- Cone $z = 2 - \sqrt{x^2 + y^2}$



Region bounded by 2 surfaces (2)

Expression as an integral: We have

$$V = \int\int_R \left(2 - \sqrt{x^2+y^2} - \left(x^2+y^2\right)\right) \mathrm{d}A,$$

Integration region: The region R is defined as

$R \equiv$ region with boundary C
given as intersection of paraboloid and cone

Region bounded by 2 surfaces (3)

Definition of C: Write

$$x^2 + y^2 = 2 - \sqrt{x^2 + y^2}$$

In polar coordinates in the plane, this gives

$$r^2 + r - 2 = 0$$

Physical solution to the equation: Circle in the xy-plane,

$$x^2 + y^2 = 1$$

Region bounded by 2 surfaces (4)

Volume in polar coordinates: We have

$$V = \int\int_R \left(2 - \sqrt{x^2+y^2} - \left(x^2+y^2\right)\right) \mathrm{d}A$$
$$= \int_0^{2\pi}\int_0^1 \left(2 - r - r^2\right) r \,\mathrm{d}r\mathrm{d}\theta$$
$$= \int_0^{2\pi} \left(r^2 - \frac{1}{3}r^3 - \frac{1}{4}r^4\right)\Big|_0^1$$

Thus

$$V = \frac{5\pi}{6}$$

Outline

Triple integral approximation (1)

Aim: For $w = f(x, y, z)$, compute
$\hookrightarrow$ The integral of f on a domain $D \subset \mathbb{R}^3$

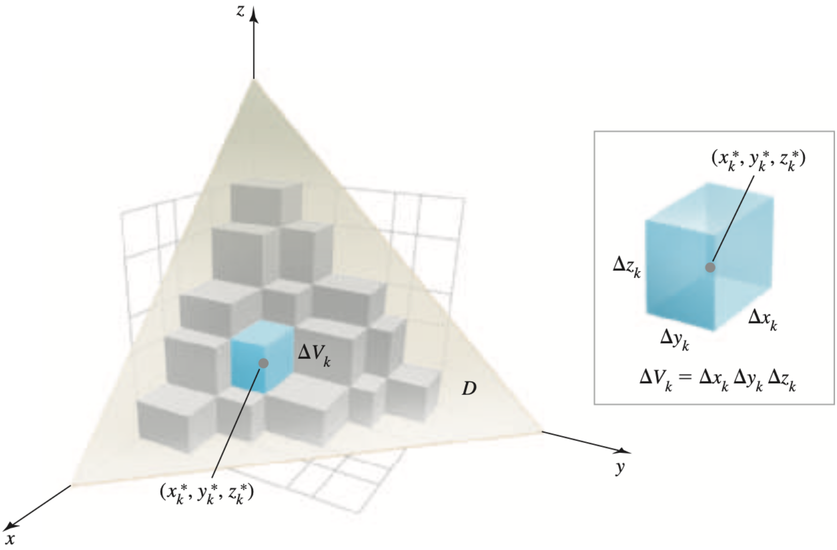
Approximation:

- Divide D into boxes centered at (x_k^*, y_k^*, z_k^*)
- Area of each box: $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$

Then we obtain the integral as a limit

$$\int\int\int_D f(x, y, z)\, \mathrm{d}V = \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\, \Delta V_k$$

Triple integral approximation (2)



Choosing the order of integration

Theorem 3.

Let

- f continuous function on $\mathbb{R}^3$
- D domain of the form

$$D = \Big\{(x, y, z);\ a \le x \le b,\ g(x) \le y \le h(x),\ G(x, y) \le z \le H(x, y)\Big\}$$

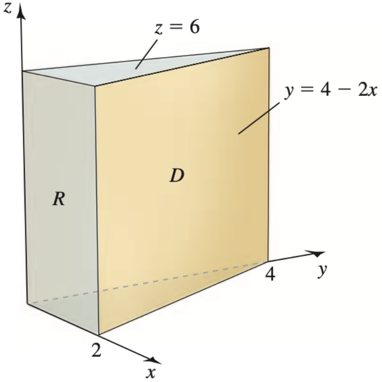
Then we have

$$\int\int\int_D f(x, y, z)\,\mathrm{d}V = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x, y, z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x$$

Volume of a prism (1)

Problem: Compute the volume of a prism D

- In the first octant
- Bounded by planes $y = 4 - 2x$ and $z = 6$



Volume of a prism (2)

Strategy of integration:

1. Upper surface: $y = 4 - 2x$
2. Base: $y = 0$, $0 \leq x \leq 2$, $0 \leq z \leq 6$
   $\hookrightarrow$ We get a rectangle (easy surface)

Conclusion: an easy way to integrate is in this order,

$$\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$$

Volume of a prism (3)

Integral computation: We get

$$V = \int_0^6 \int_0^2 \int_0^{4-2x} \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}z = \int_0^6 \int_0^2 (4-2x) \, \mathrm{d}x \, \mathrm{d}z$$

Thus we get

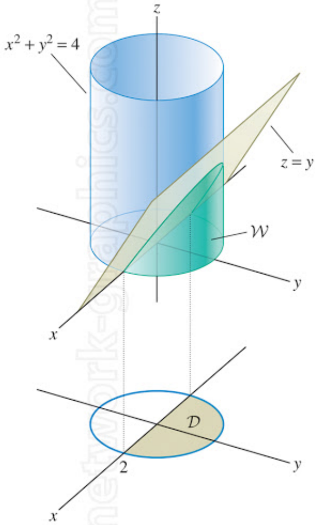
$$V = 24$$

Volume of a wedge (1)

Problem:

Compute the volume of the cylinder $C : x^2 + y^2 = 1$ delimited by

- xy-plane $z = 0$
- Plane $z = y$



Volume of a wedge (2)

Strategy of integration:

- In xy-plane, surface delimited by $x^2 + y^2 = 1$ and $y \geq 0$
  $\hookrightarrow$ Easy domain $\mathcal{D}$ (half circle)

Conclusion: an easy way to integrate is in this order,

$$\mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$$

Volume of a wedge (3)

Integral computation: We get

$$
\begin{aligned}
V &= \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} \int_{0}^{y} \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} y \, \mathrm{d}y \, \mathrm{d}x \\
&= \frac{1}{2} \int_{-1}^{1} (1 - x^2) \, \mathrm{d}x
\end{aligned}
$$

Thus we get

$$
V = \frac{2}{3}
$$

Volume between two cones (1)

Problem:

Compute the volume

- Above cone $C_1 : z = \sqrt{x^2 + y^2}$
- Below cone $C_2 : z = 2 - \sqrt{x^2 + y^2}$

Volume between two cones (2)

Intersection of the 2 cones: Its projection on xy-plane is

$$x^2 + y^2 = 1$$

Strategy of integration:

- In xy-plane, surface delimited by $x^2 + y^2 = 1$
  $\hookrightarrow$ Easy domain (circle)

Conclusion: an easy way to integrate is in this order,

$$\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

Volume between two cones (3)

Integral computation: We get

$$V = \int_{x^2+y^2\le 1}\int_{\sqrt{x^2+y^2}}^{2-\sqrt{x^2+y^2}} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_{x^2+y^2\le 1}\left(2-2\sqrt{x^2+y^2}\right)\mathrm{d}y\,\mathrm{d}x$$

Remark:
Terrible integral in Cartesian coordinates!

Volume between two cones (4)

Polar domain:

$$0 \leq r \leq 1, \qquad 0 \leq \theta \leq 2\pi$$

Volume in polar coordinates:

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 (2 - 2r)\, r \, \mathrm{d}r \, \mathrm{d}\theta \\ &= 2\pi \times \frac{1}{3} \end{aligned}$$

We get

$$V = \frac{2\pi}{3}$$

Outline

Definition of cylindrical coordinates

Notation for cylindrical coordinates: Similar to polar coordinates

$$(r, \theta, z)$$

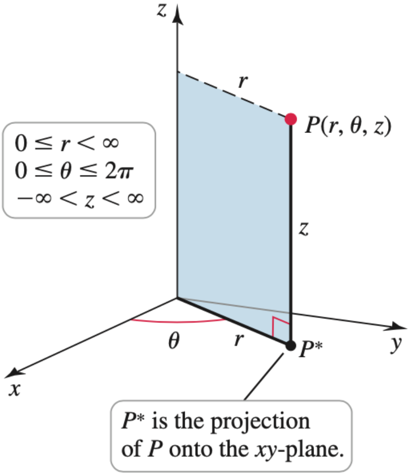
Conversion Cartesian to cylindrical:

$$r^2 = x^2 + y^2, \qquad \tan(\theta) = \frac{y}{x}, \qquad z = z$$

Conversion cylindrical to Cartesian:

$$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad z = z$$

Cylindrical coordinates: illustration



Example of cylindrical coordinates

Point in Cartesian coordinates:

$$P(-3, 3\sqrt{3}, 1)$$

Problem:

Find cylindrical coordinates for P

Answer:

$$\left(6, \frac{2\pi}{3}, 1\right)$$

Sets easily written in cylindrical coordinates

Cylinder:

$$r = a$$

Cylindrical shell:

$$a \leq r \leq b$$

Vertical half plane:

$$\theta = \theta_0$$

Horizontal plane:

$$z = a$$

Cone:

$$z = a\, r$$

Another domain in cylindrical coordinates (1)

Domain:

$$D = \left\{ (r, \theta, z);\ r^2 \le z \le 4 \right\}$$

Problem:

Identify this domain

Another domain in cylindrical coordinates (2)

Lower bound on z: Given by the surface

$$z = r^2 \quad \Longleftrightarrow \quad z = x^2 + y^2$$

This is a paraboloid

Upper bound on z: Given by the surface

$$z = 4$$

This is a horizontal plane

Integration in cylindrical coordinates

Basic formula: In cylindrical coordinates (r, θ, z),

$$\int\int\int_D f(x, y, z)\,\mathrm{d}V = \int\int\int_D f(r\cos(\theta), r\sin(\theta), z)\, r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$$

When to use cylindrical coordinates: If

1. The domain D is one of the cylinder type domains
   $\hookrightarrow$ mentioned before
2. f is a function of $x^2 + y^2$

Example of cylindrical integral (1)

Problem: Compute

$$I = \int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\, \mathrm{d}z \mathrm{d}y \mathrm{d}x$$

Preliminary remark:
Awful integral in Cartesian coordinates!

Example of cylindrical integral (2)

Domain:

$$-3 \leq x \leq 3 \quad \text{and} \quad 0 \leq y \leq \sqrt{9 - x^2}$$
$$\Longleftrightarrow$$
$$0 \leq r \leq 3 \quad \text{and} \quad 0 \leq \theta \leq \pi$$

Computing the integral: With cylindrical coordinates,

$$I = \int_0^{\pi} \int_0^3 \int_0^{9-r^2} r \, \mathrm{d}z \, r \, \mathrm{d}r \mathrm{d}\theta$$

We get

$$I = \frac{162\pi}{5}$$

Mass of a solid paraboloid (1)

Definition of the solid: Bounded by

- Paraboloid $z = 4 - r^2$
- Plane $z = 0$

Problem: Find mass of solid if density is

$$f(r, \theta, z) = 5 - z$$

Mass of a solid paraboloid (2)

Domain: We have

$$D = \left\{ 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 4 - r^2 \right\}$$

Mass: We compute

$$\begin{aligned} M &= \int\int\int_D f(r,\theta,z)\,\mathrm{d}V \\ &= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} (5-z)\,\mathrm{d}z\, r\,\mathrm{d}r\mathrm{d}\theta \\ &= \frac{1}{2}\int_0^{2\pi}\int_0^2 \left(24r - 2r^3 - r^5\right)\ r\,\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{44}{3}\,\mathrm{d}\theta = \frac{88\pi}{3} \end{aligned}$$

Definition of spherical coordinates

Notation for spherical coordinates: Similar to polar coordinates

$$(\rho, \varphi, \theta), \quad \text{with} \quad \rho \geq 0, \quad 0 \leq \varphi \leq \pi, \quad 0 \leq \theta \leq 2\pi$$

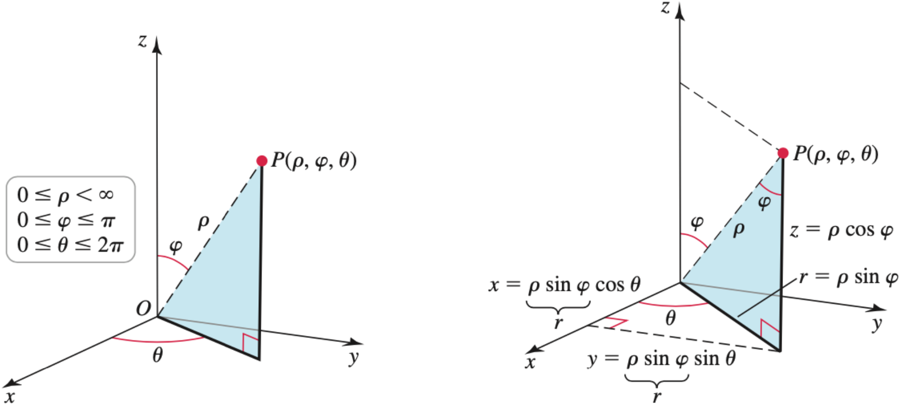
Conversion Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2, \qquad + \text{ trigonometry to find } \varphi, \theta$$

Conversion spherical to Cartesian:

$$x = \rho \sin(\varphi) \cos(\theta), \qquad y = \rho \sin(\varphi) \sin(\theta), \qquad z = \rho \cos(\varphi)$$

Spherical coordinates: illustration



Example of spherical coordinates

Point in spherical coordinates:

$$P\left(1, \frac{\pi}{6}, \frac{\pi}{3}\right)$$

Problem:

Find Cartesian coordinates for P

Answer:

$$\left(\frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$$

Sets easily written in spherical coordinates

Sphere:

$$\rho = a$$

Vertical half plane:

$$\theta = \theta_0$$

Horizontal plane:

$$\rho = a \sec(\varphi)$$

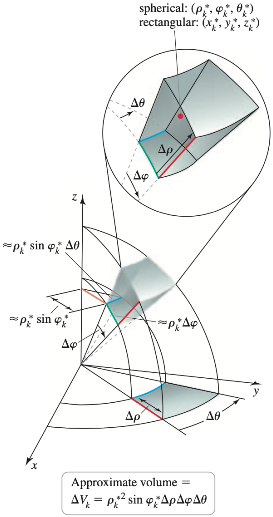
Cone:

$$\varphi = \varphi_0$$

Small spherical volume

Formula: We have

$$\mathrm{d}V = \rho^2 \sin(\varphi)\, \mathrm{d}\rho\, \mathrm{d}\theta\, \mathrm{d}\varphi$$



Integration in spherical coordinates

Basic formula: In spherical coordinates (r, θ, z),

$$\int\int\int_D f(x, y, z)\,\mathrm{d}V$$
$$= \int\int\int_D f\left(\rho\cos(\theta)\sin(\varphi), \rho\sin(\theta)\sin(\varphi), \rho\cos(\varphi)\right)\ \rho^2\sin(\varphi)\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

When to use spherical coordinates: If

1. The domain D is one of the spherical type domains
   $\hookrightarrow$ mentioned before
2. f is a function of $x^2 + y^2 + z^2$

Example of spherical integral (1)

Domain: We consider

$D =$ region in the first octant between two spheres of radius 1 and 2 centered at the origin.

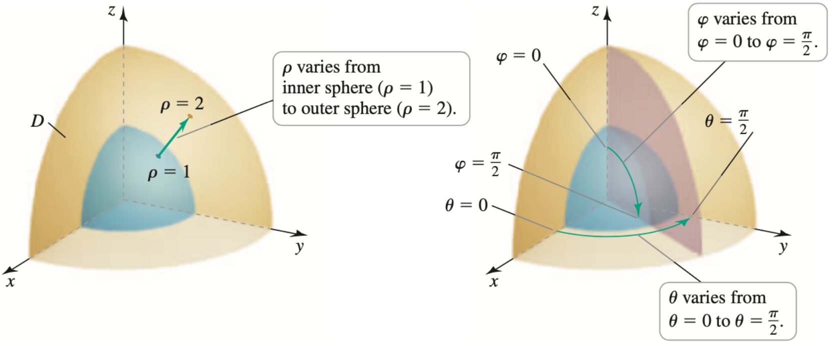
Problem: Compute

$$I = \int\int\int_D \left(x^2 + y^2 + z^2\right)^{-3/2} \mathrm{d}V$$

Example of spherical integral (2)

Expressing D in spherical coordinates:

$$D = \left\{1 \le \rho \le 2,\ 0 \le \varphi \le \frac{\pi}{2},\ 0 \le \theta \le \frac{\pi}{2}\right\}$$



Example of spherical integral (3)

Integral in spherical coordinates:

$$I = \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^{-3} \rho^2 \sin(\varphi) \, \mathrm{d}\rho \, \mathrm{d}\varphi \, \mathrm{d}\theta$$

Computation:

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^{\pi/2} \ln(\rho)\Big|_1^2 \sin(\varphi) \, \mathrm{d}\varphi \, \mathrm{d}\theta \\ &= \ln(2) \int_0^{\pi/2} (-\cos(\varphi)) \Big|_0^{\pi/2} \mathrm{d}\theta \\ &= \frac{\ln(2)\,\pi}{2} \end{aligned}$$

Volume of an ice cream cone (1)

Domain: We consider

$$D = \text{region between cone } \varphi = \frac{\pi}{6} \text{ and sphere } \rho = 4.$$

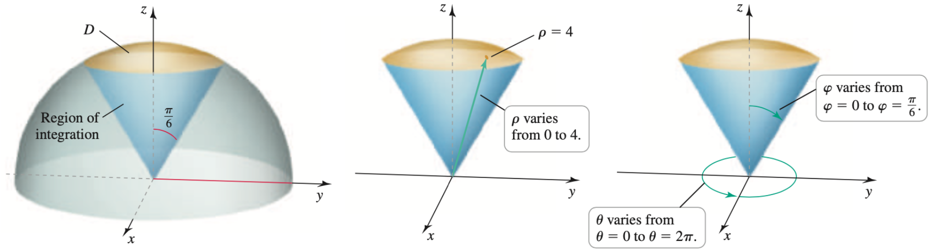
Problem: Compute

$$V = \text{Volume of } D = \int\int\int_D \mathrm{d}V$$

Volume of an ice cream cone (2)

Expressing D in spherical coordinates:

$$D = \left\{ 0 \le \rho \le 4,\ 0 \le \varphi \le \frac{\pi}{6},\ 0 \le \theta \le 2\pi \right\}$$



Volume of an ice cream cone (3)

Integral in spherical coordinates:

$$I = \int_0^{2\pi} \int_0^{\pi/6} \int_0^4 \rho^2 \sin(\varphi)\, \mathrm{d}\rho\, \mathrm{d}\varphi\, \mathrm{d}\theta$$

Computation:

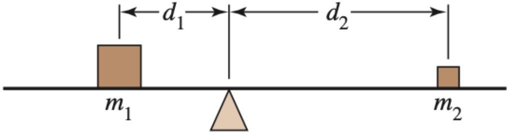
$$\begin{aligned} I &= \int_0^{2\pi} \int_0^{\pi/6} \frac{\rho^3}{3}\Big|_0^4 \sin(\varphi)\, \mathrm{d}\varphi\, \mathrm{d}\theta \\ &= \frac{64}{3} \int_0^{2\pi} \left(-\cos(\varphi)\right)\Big|_0^{\pi/6} \mathrm{d}\theta \\ &= \frac{64}{3}\left(1 - \frac{\sqrt{3}}{2}\right) 2\pi \\ &= \frac{64\pi(2-\sqrt{3})}{3} \end{aligned}$$

Outline

A playground example (1)

Seesaw principle: Seesaw in equilibrium if

$$m_1 d_1 = m_2 d_2$$

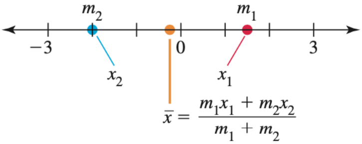


A playground example (2)

Notation:
Call $\bar{x}$ the center of mass for the 2-body seesaw system

Seesaw principle revisited: Seesaw in equilibrium if

$$m_1(x_1 - \bar{x}) = m_2(\bar{x} - x_2)$$



Solving for $\bar{x}$: We get

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{\text{Average(mass} \times \text{distance)}}{\text{Average(mass)}}$$

Center of mass of a 3-d body

Theorem 4.

Let

- D closed bounded region in $\mathbb{R}^3$
- $\rho =$ Density function on D
- Mass of D given by $m = \int\int\int_D \rho(x,y,z)\,\mathrm{d}V$

Then the coordinates of center of mass for D are

$$\begin{aligned}
\bar{x} &= \frac{M_{yz}}{m} = \frac{1}{m}\int\int\int_D x\,\rho(x,y,z)\,\mathrm{d}V \\
\bar{y} &= \frac{M_{xz}}{m} = \frac{1}{m}\int\int\int_D y\,\rho(x,y,z)\,\mathrm{d}V \\
\bar{z} &= \frac{M_{xy}}{m} = \frac{1}{m}\int\int\int_D z\,\rho(x,y,z)\,\mathrm{d}V
\end{aligned}$$

Moments

Definition of moment: In center of mass definition, the quantity

$$M_{xy} = \int\int\int_D z\,\rho(x,y,z)\,\mathrm{d}V$$

is called moment with respect to the xy-plane.

Remark: Moments are of the form

$$\text{Average}(\text{mass} \times \text{distance})$$

A 2-d example (1)

Domain: We consider $D \subset \mathbb{R}^2$ defined by

$$R = \left\{(x,y); 1 \leq x^2 + y^2 \leq 4\right\} \bigcap \text{ First quadrant}$$

Density of mass: Given by

$$\rho(x,y) = \sqrt{x^2 + y^2}$$

Problem:

Find the center of mass of this object

A 2-d example (2)

Total mass: We get (with convenient polar coordinates)

$$\begin{aligned} m &= \int\int_R \rho \, \mathrm{d}A \\ &= \int_0^{\pi/2}\int_1^2 \rho \, r \, \mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{\pi/2}\int_1^2 r^2 \, \mathrm{d}r\mathrm{d}\theta \end{aligned}$$

Thus

$$m = \frac{7\pi}{6}$$

A 2-d example (3)

Center of mass on the y-axis: We have

$$
\begin{aligned}
\bar{y} &= \frac{1}{m}\int\int_R y\,\rho\,\mathrm{d}A \\
&= \frac{6\pi}{7}\int_0^{\pi/2}\int_1^2 r\sin(\theta)\rho\,r\,\mathrm{d}r\mathrm{d}\theta \\
&= \frac{6\pi}{7}\int_0^{\pi/2}\int_1^2 r^3\,\sin(\theta)\,\mathrm{d}r\mathrm{d}\theta \\
&= \frac{45}{14\pi}
\end{aligned}
$$

Thus

$$\bar{y} \simeq 1.023$$

A 2-d example (4)

Center of mass on the x-axis: We have

$$\begin{aligned}
\bar{x} &= \frac{1}{m}\int\int_R x\,\rho\,\mathrm{d}A \\
&= \frac{6\pi}{7}\int_0^{\pi/2}\int_1^2 r\cos(\theta)\rho\,r\,\mathrm{d}r\mathrm{d}\theta \\
&= \frac{6\pi}{7}\int_0^{\pi/2}\int_1^2 r^3\,\cos(\theta)\,\mathrm{d}r\mathrm{d}\theta \\
&= \frac{45}{14\pi}
\end{aligned}$$

Thus

$$\bar{x} \simeq 1.023$$

A 2-d example (5)

Conclusion: The center of mass is

$$(\bar{x}, \bar{y}) = (1.023, 1.023)$$

A 3-d example (1)

Domain: We consider $D \subset \mathbb{R}^3$ bounded by

- Hemisphere with radius a
- xy-plane

Density of mass: Given by (object heavier close to the center)

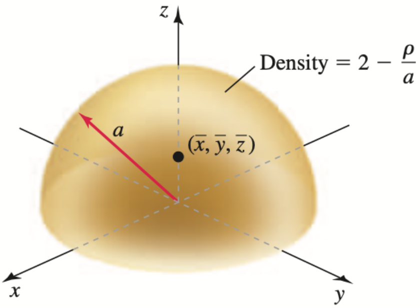
$$f(\rho, \varphi, \theta) = 2 - \frac{\rho}{a}$$

Problem:

Find the center of mass of this object

A 3-d example (2)

Graph of the situation:



A 3-d example (3)

Total mass: We get (with convenient spherical coordinates)

$$
\begin{aligned}
m &= \int\int_D f \, \mathrm{d}V \\
&= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a \left(2 - \frac{\rho}{a}\right) \rho^2 \sin(\varphi) \, \mathrm{d}\rho \mathrm{d}\varphi \mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} \left(\frac{2\rho^3}{3} - \frac{\rho^4}{4a}\right)\Big|_0^a \sin(\varphi) \, \mathrm{d}\rho \mathrm{d}\varphi \mathrm{d}\theta \\
&= \frac{5a^3}{12}\int_0^{2\pi}\int_0^{\pi/2} \sin(\varphi) \, \mathrm{d}\rho \mathrm{d}\varphi \mathrm{d}\theta \\
&= \frac{5a^3}{12} \times 2\pi
\end{aligned}
$$

Thus

$$
m = \frac{5\pi a^3}{6}
$$

A 3-d example (4)

Moment wrt the xy-axis: We have

$$\begin{aligned} M_{xy} &= \int\int_D z\, f \,\mathrm{d}V \\ &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho\,\cos(\varphi)\left(2-\frac{\rho}{a}\right)\,\rho^2\sin(\varphi)\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta \\ &= \int_0^{2\pi}\int_0^{\pi/2}\left(\frac{\rho^4}{2}-\frac{\rho^5}{5a}\right)\Big|_0^a \cos(\varphi)\,\sin(\varphi)\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta \\ &= \frac{3a^4}{10}\int_0^{2\pi}\int_0^{\pi/2}\frac{1}{2}\sin(2\varphi)\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta \\ &= \frac{3a^4}{10}\times\frac{1}{2}\times 2\pi \end{aligned}$$

Thus

$$M_{xy} = \frac{3\pi a^4}{10}$$

A 3-d example (5)

Center of mass on the z-axis: We have

$$\bar{z} = \frac{M_{xy}}{m} = \frac{3\pi a^4/10}{5\pi a^3/6}$$

Thus

$$\bar{z} = \frac{9a}{25} = 0.36a$$

Remark: For a uniform half sphere we would find
$\hookrightarrow \bar{z} = 0.375a$