Vector calculus

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Multivariate calculus - MA 261

Mostly taken from *Calculus, Early Transcendentals*
by Briggs - Cochran - Gillett - Schulz

Outline

Outline

Definition of vector field

Multivariate function: Recall that

- $z = f(x, y)$ was a function of 2 variables
- For each (x, y), $z \in \mathbb{R}$
- This is called a scalar field

Vector field in $\mathbb{R}^2$:

- Of the form $\mathbf{F}(x, y) = \langle f(x, y), g(x, y) \rangle$
- For each (x, y), $\mathbf{F} \in \mathbb{R}^2$, namely $\mathbf{F}$ is a vector

Example of vector field

Definition of the vector field:

$$\mathbf{F}(x, y) = \langle x, y \rangle$$

Examples of values:

$$\begin{aligned} \mathbf{F}(1,1) &= \langle 1,1 \rangle \\ \mathbf{F}(0,2) &= \langle 0,2 \rangle \\ \mathbf{F}(-1,-2) &= \langle -1,-2 \rangle \end{aligned}$$

Shear field (1)

Definition of the vector field:

$$\mathbf{F}(x, y) = \langle 0, x \rangle$$

Problem:
Give a representation of **F**

Shear field (2)

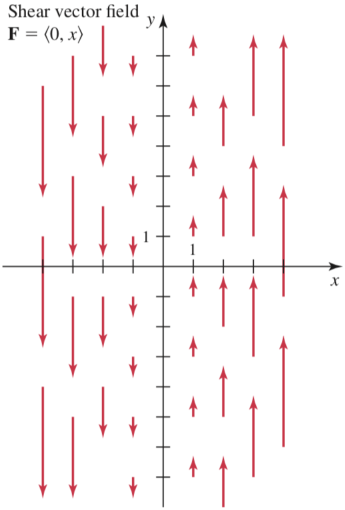
Recall:

$$\mathbf{F}(x, y) = \langle 0, x \rangle$$

Information about the vector field:

1. $\mathbf{F}(x, y)$ independent of y
2. $\mathbf{F}(x, y)$ points in the y direction
3. If $x > 0$, $\mathbf{F}(x, y)$ points upward
4. If $x < 0$, $\mathbf{F}(x, y)$ points downward
5. Magnitude of $\mathbf{F}(x, y)$ gets larger
   $\hookrightarrow$ as we move away from the origin

Shear field (3)



Rotation field (1)

Definition fo the vector field:

$$\mathbf{F}(x, y) = \langle -y, x \rangle$$

Problem:

Give a representation of **F**

Rotation field (2)

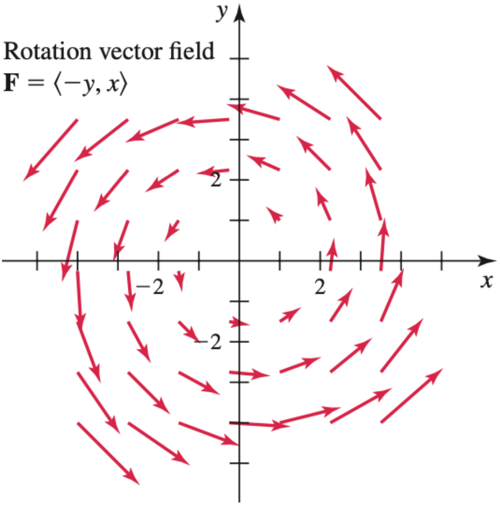
Recall:

$$\mathbf{F}(x, y) = \langle -y, x \rangle$$

Information about the vector field:

1. Magnitude increases as $x \to \infty$ or $y \to \infty$
2. If $y = 0$ and $x > 0$, $\mathbf{F}(x, y)$ points upward
3. If $y = 0$ and $x < 0$, $\mathbf{F}(x, y)$ points downward
4. If $x = 0$ and $y > 0$, $\mathbf{F}(x, y)$ points in negative x direction
5. If $x = 0$ and $y < 0$, $\mathbf{F}(x, y)$ points in positive x direction
6. Draw a few more points
   $\hookrightarrow$ We get a rotation field

Rotation field (3)



Radial vector fields

Definition 1.

We set

$$\mathbf{r} = \langle x, y \rangle$$

Then

General definition: A radial vector field is of the form

$$\mathbf{F} = f(x, y)\,\mathbf{r}, \quad \text{with} \quad f(x, y) \in \mathbb{R}$$

Fields of special interest:

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p}$$

Normal and tangent vectors (1)

Situation: We consider

- Function $g(x, y) = x^2 + y^2$
- Circle $C : \{(x, y);\, g(x, y) = a^2\}$
- Field $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|}$

Problem: For $(x, y) \in C$, prove that

$$\mathbf{F}(x, y) \perp \text{tangent line to } C \text{ at } (x, y)$$

Normal and tangent vectors (2)

Recall: From level curves considerations, we have

$$\nabla g(x, y) \perp \text{tangent line to } C \text{ at } (x, y)$$

Computing the gradient: We get

$$\langle 2x, 2y \rangle \perp \text{tangent line to } C \text{ at } (x, y)$$

Conclusion: Since $\langle 2x, 2y \rangle = 2\mathbf{r}$, we end up with

$$\mathbf{r} \perp \text{tangent line to } C \text{ at } (x, y)$$

Vector field in $\mathbb{R}^3$

Definition of vector fields in $\mathbb{R}^3$:

- Of the form $\mathbf{F}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle$
- For each (x, y, z), $\mathbf{F} \in \mathbb{R}^3$, namely $\mathbf{F}$ is a vector

Radial vector fields: Of the form

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y, z \rangle}{|\mathbf{r}|^p}$$

Example of vector field in $\mathbb{R}^3$ (1)

Definition of the vector field:

$$\mathbf{F}(x, y, z) = \left\langle x, y, e^{-z} \right\rangle$$

Problem:
Give a representation of **F**

Example of vector field in $\mathbb{R}^3$ (2)

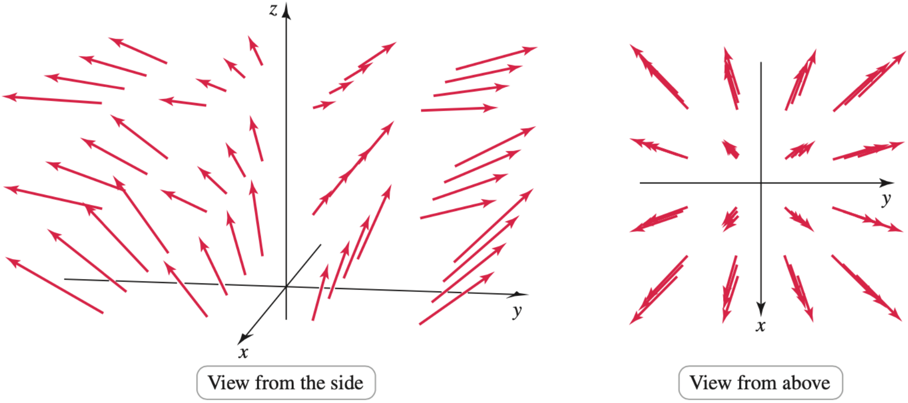
Recall:

$$\mathbf{F}(x,y) = \left\langle x, y, e^{-z} \right\rangle$$

Information about the vector field:

1. xy-trace: $\mathbf{F} = \langle x, y, 1 \rangle$
   $\hookrightarrow$ Radial in the plane, with component 1 in vertical direction
2. In horizontal plane $z = z_0$: $\mathbf{F} = \langle x, y, e^{-z_0} \rangle$
   $\hookrightarrow$ Radial in the plane, with smaller component in vert. direction
3. As $z \to \infty$: $\mathbf{F} \to \langle x, y, 0 \rangle$
   $\hookrightarrow$ Radial in the plane, with 0 component in vertical direction
4. Magnitude increases as we move away from vertical axis

Example of vector field in $\mathbb{R}^3$ (3)



Outline

Motivation

Physical situation: Assume we want to compute

- Work of gravitational field $\mathbf{F}$
- Along the (curved) path C of a satellite

Needed quantity: integral of $\mathbf{F}$ along C
$\hookrightarrow$ How to compute that?

Approximation procedure

Notation: We consider

- Curve $\mathbf{r}(t) = \langle x(t), y(t) \rangle$
- Partition $a = t_0 < \cdots < t_n = b$ of time interval $[a, b]$
- Arc length s of $\mathbf{r}$
- Function f defined on $\mathbb{R}^2$

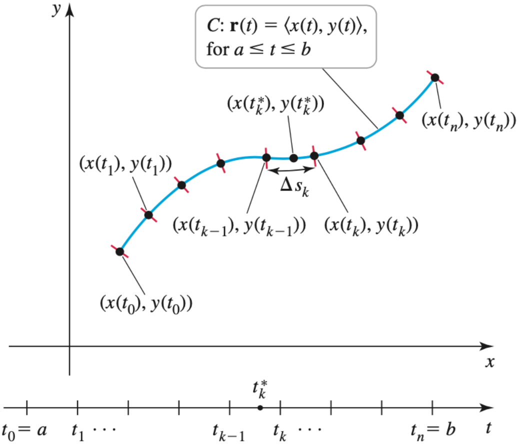
Approximation:

$$S_n = \sum_{k=1}^{n} f\left(x(t_k), y(t_k)\right) \Delta s_k$$

Approximation procedure: illustration

Recall:

$$S_n = \sum_{k=1}^{n} f\left(x(t_k), y(t_k)\right) \Delta s_k$$



Computation of line integrals in $\mathbb{R}^2$

Theorem 2.

We consider

- Curve C defined by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$
- Time interval $[a, b]$
- Arc length s of $\mathbf{r}$
- Function f defined on $\mathbb{R}^2$

Then we have

$$\int_C f \, \mathrm{d}s = \int_a^b f\left(x(t), y(t)\right) |\mathbf{r}'(t)| \mathrm{d}t$$

Computation of line integrals

Recipe:

1. Find parametric description of C
   $\hookrightarrow \mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $t \in [a, b]$
2. Compute $|\mathbf{r}'(t)| = \sqrt{x^2(t) + y^2(t)}$
3. Make substitutions for x and y and evaluate ordinary integral

$$\int_a^b f\left(x(t), y(t)\right) |\mathbf{r}'(t)| \mathrm{d}t$$

Average temperature (1)

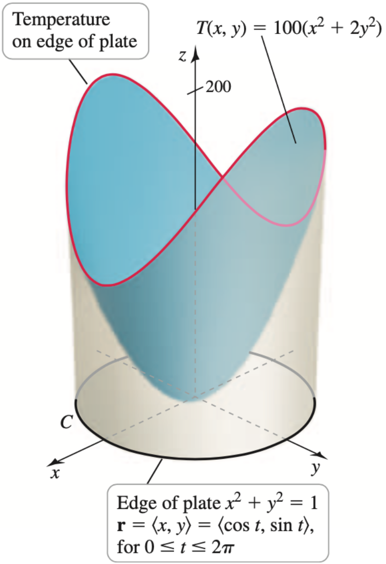
Situation:

- Circular plate
$$R = \left\{x^2 + y^2 = 1\right\}$$
- Temperature distribution in the plane:
$$T(x, y) = 100\left(x^2 + 2y^2\right)$$

Problem:

Compute the average temperature on the edge of the plate

Average temperature (2)



Average temperature (3)

Parametric description of C: $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$

Arc length: $|\mathbf{r}'(t)| = 1$

Line integral:

$$\begin{aligned}\int_C T(x,y)\,\mathrm{d}s &= 100 \int_0^{2\pi} \left(x(t)^2 + 2y(t)^2 \right) |\mathbf{r}'(t)|\,\mathrm{d}t \\ &= 100 \int_0^{2\pi} \left(\cos^2(t) + 2\sin^2(t) \right)\,\mathrm{d}t \\ &= 100 \int_0^{2\pi} \left(1 + \sin^2(t) \right)\,\mathrm{d}t\end{aligned}$$

Thus

$$\int_C T(x,y)\,\mathrm{d}s = 300\pi$$

Average temperature (4)

Recall:

$$\int_C T(x,y)\,\mathrm{d}s = 300\pi$$

Average temperature: Given by

$$\overline{T} = \frac{\int_C T(x,y)\,\mathrm{d}s}{\text{Length}(C)}$$

We get

$$\overline{T} = \frac{300\pi}{2\pi} = 150$$

Computation of line integrals in $\mathbb{R}^3$

Theorem 3.

We consider

- Curve C defined by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$
- Time interval $[a, b]$
- Arc length s of $\mathbf{r}$
- Function f defined on $\mathbb{R}^3$

Then we have

$$\int_C f \, \mathrm{d}s = \int_a^b f\,(x(t), y(t), z(t))\; |\mathbf{r}'(t)| \mathrm{d}t$$

Example of line integral in $\mathbb{R}^3$ (1)

Situation:

- Two points in $\mathbb{R}^3$

$$P(1,0,0), \qquad Q(0,1,1)$$

- Function:

$$f(x,y,z) = xy + 2z$$

Problem: Compute $\int_C f(x,y)\,\mathrm{d}s$ in the following cases:

1. C is the segment from P to Q
2. C is the segment from Q to P

Example of line integral in $\mathbb{R}^3$ (2)

Parametric equation for segment from P to Q:

$$\mathbf{r}(t) = \langle 1 - t, t, t \rangle \,, \qquad t \in [0, 1]$$

Arc length:

$$|\mathbf{r}'(t)| = \sqrt{3}$$

Example of line integral in $\mathbb{R}^3$ (3)

Line integral:

$$\begin{aligned}\int_C f(x,y)\,\mathrm{d}s &= \int_C (xy + 2z)\;\mathrm{d}s \\ &= \int_0^1 ((1-t)t + 2t)\,\sqrt{3}\,\mathrm{d}t \\ &= \sqrt{3}\int_0^1 \left(3t - t^2\right)\;\mathrm{d}t \\ &= \sqrt{3}\left(\frac{3}{2} - \frac{1}{3}\right)\end{aligned}$$

Thus we get

$$\int_C f(x,y)\,\mathrm{d}s = \frac{7\sqrt{3}}{6}$$

Example of line integral in $\mathbb{R}^3$ (4)

Parametric equation for segment from Q to P:

$$\mathbf{r}(t) = \langle t, 1-t, 1-t \rangle$$

Arc length: $|\mathbf{r}'(t)| = \sqrt{3}$

Line integral: One can check that we also have

$$\int_C f(x,y)\,\mathrm{d}s = \frac{7\sqrt{3}}{6}$$

General conclusion:

The value of $\int_C f(x,y)\,\mathrm{d}s$
does not depend on the parametrization of C

Line integral of a vector field

Definition 4.

We consider

- Curve $C : \mathbf{r}(s) = \langle x(s), y(s), z(s) \rangle$
- C is parametrized by arc length s
- $\mathbf{T}(s)$ unit tangent vector
- Vector field $\mathbf{F}$ defined on $\mathbb{R}^3$

Then the line integral of $\mathbf{F}$ over C is

$$\int_C \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s$$

Motivation: Line integrals crucial to compute work of a force $\mathbf{F}$

Computing line integrals

Theorem 5.

We consider

- Curve $C : \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$
- C is parametrized by $t \in [a, b]$
- Vector field $\mathbf{F}$ defined on $\mathbb{R}^3$

Then the line integral of $\mathbf{F}$ over C is given by

$$\int_C \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s = \int_C \mathbf{F}(t) \cdot \mathbf{r}'(t) \, \mathrm{d}t$$

Example of line integral for a vector field (1)

Situation:

- Two points in $\mathbb{R}^2$:

$$P(0,1), \qquad Q(1,0)$$

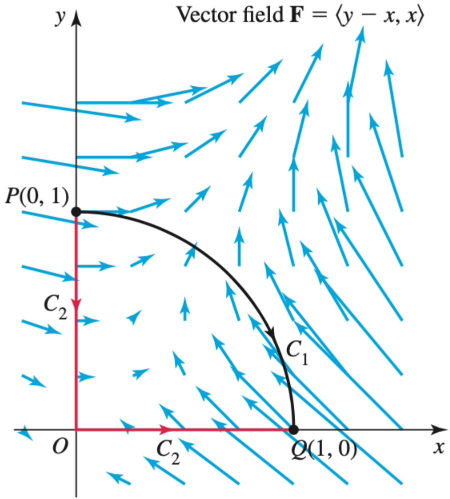
- Vector field:

$$\mathbf{F}(x,y) = \langle y - x,\, x \rangle$$

Problem: Compute $\int_C \mathbf{F} \cdot \mathbf{T}\, \mathrm{d}s$ in the following cases:

1. C_1 quarter-circle from P to Q
2. $-C_1$ quarter-circle from Q to P
3. C_2 path defined by segments $P(0,1)$–$O(0,0)$–$Q(1,0)$

Example of line integral for a vector field (2)



Example of line integral for a vector field (3)

Parametric equation for C_1:

$$\mathbf{r}(t) = \langle \sin(t), \cos(t) \rangle$$

Parametric equation for $\mathbf{F}$: Along C_1 we have

$$\mathbf{F} = \langle y - x,\ x \rangle = \langle \cos(t) - \sin(t), \sin(t) \rangle$$

Dot product: We have

$$\mathbf{F}(t) \cdot \mathbf{r}'(t) = \cos^2(t) - \sin^2(t) - \sin(t)\cos(t) = \cos(2t) - \frac{1}{2}\sin(2t)$$

Example of line integral for a vector field (4)

Line integral:

$$\int_{C_1} \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s = \int_{C_1} \mathbf{F}(t) \cdot \mathbf{r}'(t) \, \mathrm{d}t$$
$$= \int_0^{\pi/2} \left(\cos(2t) - \frac{1}{2} \sin(2t) \right) \, \mathrm{d}t$$
$$= \frac{1}{2} \sin(2t) + \frac{1}{4} \cos(2t) \Big|_0^{\pi/2}$$

Thus we get

$$\int_{C_1} \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s = -\frac{1}{2}$$

Example of line integral for a vector field (5)

Line integral along $-C_1$: We find

$$\int_{-C_1} \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s = \frac{1}{2} = -\int_{C_1} \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s$$

Changing the orientation of C_1 changes the sign of the line integral

Line integral along C_2: We find

$$\int_{C_2} \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s = -\frac{1}{2} = \int_{C_1} \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s$$

Question: is this true for a large class of **F**?

Outline

Main issues in this section

Two important questions:

1. When can we say that a vector field is the gradient of a function?
2. What is special with this kind of vector fields?

Conservative vector field

Definition 6.

Let

- D domain of $\mathbb{R}^2$
- $\mathbf{F}$ vector field defined on D

Then $\mathbf{F}$ is a conservative vector field if

$$\text{There exists } \varphi \text{ such that } \mathbf{F} = \nabla\varphi \text{ on } D$$

Criterion for being conservative in $\mathbb{R}^2$

Notation: For $\varphi : \mathbb{R}^2 \to \mathbb{R}$, set $\varphi_x = \frac{\partial \varphi}{\partial x}$ and $\varphi_y = \frac{\partial \varphi}{\partial y}$

Theorem 7.

Consider a vector field in $R \subset \mathbb{R}^2$:

$$\mathbf{F} = \langle f, g \rangle$$

Then there exists φ such that:

$$\nabla \varphi \equiv \langle \varphi_x, \varphi_y \rangle = \mathbf{F} \quad \text{on } R,$$

if and only if $\mathbf{F}$ satisfies:

$$f_y = g_x \quad \text{on } R$$

Computation of function φ in $\mathbb{R}^2$

Aim: If $f_y = g_x$, find φ such that $\varphi_x = f$ and $\varphi_y = g$.

Recipe in order to get φ:

1. Write φ as antiderivative of f with respect to x:

$$\varphi(x, y) = a(x, y) + b(y), \quad \text{where} \quad a(x, y) = \int f(x, y)\, dx$$

2. Get an equation for b by differentiating with respect to y:

$$\varphi_y = g \quad \Longleftrightarrow \quad b'(y) = g(x, y) - a_y(x, y)$$

3. Finally we get:

$$\varphi(x, y) = a(x, y) + b(y).$$

Example of conservative vector field (1)

Vector field:

$$\mathbf{F} = \langle x + y, x \rangle$$

Problem:

1. Is **F** conservative?
2. If **F** is conservative, compute φ such that $\nabla\varphi = \mathbf{F}$

Example of conservative vector field (2)

Recall:

$$\mathbf{F} = \langle x + y, x \rangle$$

Proof that **F** is conservative:

$$f_y = 1 = g_x$$

Thus **F** is conservative

Example of conservative vector field (3)

Computing φ: We have

$$\varphi = \int f(x,y)\,\mathrm{d}x + b(y) = \frac{1}{2}x^2 + yx + b(y)$$

Computing b: We write

$$\varphi_y = x \quad \Longleftrightarrow \quad x + b'(y) = x \quad \Longleftrightarrow \quad b'(y) = 0$$

Expression for φ: Since $b(y) = c$ for a constant c, we get

$$\varphi(x,y) = \frac{1}{2}x^2 + yx + c$$

Another example of conservative vector field (1)

Vector field:

$$\mathbf{F} = \langle e^x \cos(y), -e^x \sin(y) \rangle$$

Problem:

1. Is $\mathbf{F}$ conservative?
2. If $\mathbf{F}$ is conservative, compute φ such that $\nabla\varphi = \mathbf{F}$

Another example of conservative vector field (2)

Recall:

$$\mathbf{F} = \langle e^x \cos(y), -e^x \sin(y) \rangle$$

Proof that $\mathbf{F}$ is conservative:

$$f_y = -e^x \sin(y) = g_x$$

Thus $\mathbf{F}$ is conservative

Another example of conservative vector field (3)

Computing φ: We have

$$\varphi = \int f(x, y)\,\mathrm{d}x + b(y) = e^x \cos(y) + b(y)$$

Computing b: We write

$$\begin{aligned} \varphi_y = -e^x \sin(y) &\iff -e^x \sin(y) + b'(y) = -e^x \sin(y) \\ &\iff b'(y) = 0 \end{aligned}$$

Expression for φ: Since $b(y) = c$ for a constant c, we get

$$\varphi(x, y) = -e^x \sin(y) + c$$

Criterion for being conservative in $\mathbb{R}^3$

Theorem 8.

Consider a vector field in $R \subset \mathbb{R}^3$:

$$\mathbf{F} = \langle f, g, h \rangle$$

Then there exists φ such that:

$$\nabla\varphi \equiv \langle \varphi_x,\, \varphi_y,\, \varphi_z \rangle = \mathbf{F} \quad \text{on } R,$$

if and only if **F** satisfies:

$$f_y = g_x, \quad f_z = h_x, \quad g_z = h_y \quad \text{on } R$$

Computation of function φ in $\mathbb{R}^3$

Aim: If $\mathbf{F}$ is conservative, find φ
$\hookrightarrow$ such that $\varphi_x = f$, $\varphi_y = g$ and $\varphi_z = h$.

Recipe in order to get φ:

1. Write φ as antiderivative of f with respect to x:

$$\varphi(x, y) = a(x, y, z) + b(y, z), \quad \text{where} \quad a(x, y, z) = \int f(x, y, z)\, dx$$

2. Get an equation for b by differentiating with respect to y:

$$\varphi_y = g \quad \Longleftrightarrow \quad b_y(y, z) = g(x, y, z) - a_y(x, y, z)$$

3. Iterate this procedure with ∂_z

Example of conservative vector field in $\mathbb{R}^3$ (1)

Vector field:

$$\mathbf{F} = \left\langle x^2 - z\, e^y, y^3 - xz\, e^y, z^4 - x\, e^y \right\rangle$$

Problem:

1. Is $\mathbf{F}$ conservative?
2. If $\mathbf{F}$ is conservative, compute φ such that $\nabla\varphi = \mathbf{F}$

Example of conservative vector field in $\mathbb{R}^3$ (2)

Recall:

$$\mathbf{F} = \left\langle x^2 - z\, e^y, y^3 - xz\, e^y, z^4 - x\, e^y \right\rangle$$

Proof that **F** is conservative:

$$\begin{aligned} f_y = g_x &= -x\, e^y \\ f_z = h_x &= -e^y \\ g_z = h_y &= -x\, e^y \end{aligned}$$

Thus **F** is conservative

Example of conservative vector field in $\mathbb{R}^3$ (3)

Computing φ: We have

$$\varphi = \int f(x,y,z)\,\mathrm{d}x + b(y,z) = \frac{1}{3}x^3 - xz\,e^y + b(y,z)$$

Computing b: We write

$$\begin{aligned}\varphi_y = y^3 - xz\,e^y \quad &\Longleftrightarrow \quad -xz\,e^y + b_y = y^3 - xz\,e^y \\ &\Longleftrightarrow \quad b(y,z) = \frac{1}{4}y^4 + c(z)\end{aligned}$$

We have thus obtained

$$\varphi = \frac{1}{3}x^3 - xz\,e^y + \frac{1}{4}y^4 + c(z)$$

Example of conservative vector field in $\mathbb{R}^3$ (4)

Computing c: We write

$$\varphi_z = z^4 - x\,e^y \iff -x\,e^y + c'(z) = z^4 - x\,e^y$$
$$\iff c(z) = \frac{1}{5} z^5 + d$$

Expression for φ: For a constant d, we get

$$\varphi(x, y, z) = \frac{1}{3} x^3 - xz\,e^y + \frac{1}{4} y^4 + \frac{1}{5} z^5 + d$$

Fundamental theorem for line integrals

Theorem 9.

Consider

- A conservative vector field $\mathbf{F}$ on $R \subset \mathbb{R}^3$
- φ such that $\nabla\varphi = \mathbf{F}$
- A piecewise smooth oriented curve $C \subset R$ from A to B

Then we have

$$\int_C \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \varphi(B) - \varphi(A)$$

Verifying path independence (1)

Vector field:

$$\mathbf{F} = \langle x, -y \rangle$$

Curves: We consider

- C_1 quarter circle $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, \pi/2]$
- C_2 line $\mathbf{r}(t) = \langle 1 - t,\ t \rangle$ for $t \in [0, 1]$
- Both C_1 and C_2 go from $A(1, 0)$ to $B(0, 1)$

Problem:

1. Compute $\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ directly
2. Compute $\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$
   $\hookrightarrow$ using the fundamental theorem for line integrals

Verifying path independence (2)

Computation along C_1: We have

$$\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle, \qquad \mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle$$

Thus

$$\begin{aligned} \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^{\pi/2} \langle \cos(t), -\sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \, \mathrm{d}t \\ &= \int_0^{\pi/2} (-\sin(2t)) \, \mathrm{d}t \end{aligned}$$

We get

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -1$$

Verifying path independence (3)

Computation along C_2: We have

$$\mathbf{r}(t) = \langle 1-t,\, t\rangle, \qquad \mathbf{r}'(t) = \langle -1,\, 1\rangle$$

Thus

$$\begin{aligned}\int_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 \langle 1-t,\, t\rangle \cdot \langle -1,\, 1\rangle \,\mathrm{d}t \\ &= \int_0^1 (-1)\,\mathrm{d}t\end{aligned}$$

We get

$$\int_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = -1$$

Verifying path independence (4)

Computing the potential φ: We have

$$\varphi(x,y) = \frac{1}{2}\left(x^2 - y^2\right) \quad \Longrightarrow \quad \nabla\varphi = \mathbf{F}$$

Using the fundamental theorem for line integrals: We have

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \varphi(0,1) - \varphi(1,0)$$

Thus we get

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -1$$

Outline

2-dimensional curl

Definition 10.

Let

- $\mathbf{F} = \langle f, g \rangle$ vector field in $\mathbb{R}^2$

Then we define

$$\mathrm{Curl}(\mathbf{F}) = g_x - f_y$$

Another notation: In order to prepare the $\mathbb{R}^3$ version one can write

$$\mathrm{Curl}(\mathbf{F}) = (g_x - f_y)\,\mathbf{k}$$

Interpretation: $\mathrm{Curl}(\mathbf{F})$ represents
$\hookrightarrow$ The amount of rotation in $\mathbf{F}$

Example of irrotational vector field

Vector field: **F** defined by

$$\mathbf{F} = \langle x, \, y \rangle$$

Curl of **F**: We get

$$\mathrm{Curl}(\mathbf{F}) = g_x - f_y = 0$$

Interpretation: **F** has no rotational component
$\hookrightarrow$ **F** is said to be irrotational

Remark: Generally speaking, we have

$$\mathbf{F} \text{ conservative} \quad \Longrightarrow \quad \mathbf{F} \text{ irrotational}$$

Example of vector field with rotation

Vector field: **F** defined by

$$\mathbf{F} = \langle y, \, -x \rangle$$

Curl of **F**: We get

$$\mathrm{Curl}(\mathbf{F}) = g_x - f_y = -2$$

Interpretation:

F has a rotational component

Types of curves

Definition 11.

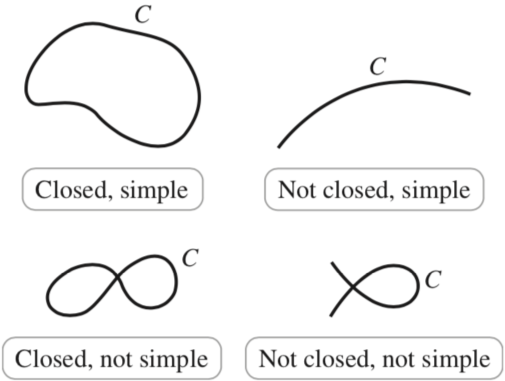
Let

- Curve $C : [a, b] \to \mathbb{R}^2$
- C given as $\mathbf{r}(t)$

Then

1. C is a simple curve if
   $\hookrightarrow r(t_1) \neq r(t_2)$ whenever $a < t_1 < t_2 < b$
2. C is a closed curve if
   $\hookrightarrow r(a) = r(b)$

Simple and closed curves



Types of domains

Definition 12.

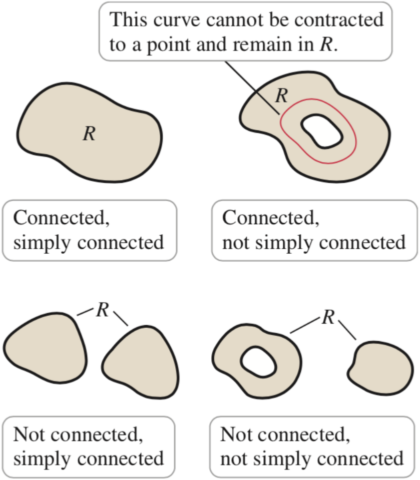
Let

- D domain of $\mathbb{R}^2$

Then

1. D is a connected domain if
   $\hookrightarrow$ it is possible to connect any two points of D by a continuous curve lying in D
2. C is a simply connected domain if
   $\hookrightarrow$ every closed simple curve in D can be deformed and contracted to a point in D

Connected and simply connected domains



General assumptions

Hypothesis for this section:

- All curves C are closed and simple
  $\hookrightarrow$ In counterclockwise direction
- All domains R are connected and simply connected

Green's theorem

Theorem 13.

Let

- $\mathbf{F} = \langle f, g \rangle$ vector field in $\mathbb{R}^2$
- C simple closed curve, counterclockwise
- C delimits a region R

Then we have

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int\int_R \mathrm{Curl}(\mathbf{F}) \, \mathrm{d}A \tag{1}$$

George Green

Some facts about Green:

- Lifespan: 1793-1841, in England
- Self taught in Math, originally a baker
- Mathematician, Physicist
- 1st mathematical theory of electromagnetism
- Went to college when he was 40
- Died 1 year later (alcoholism?)

Interpretation of Green's theorem

Interpretation of the integral on C:

- $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is a circulation integral along the boundary C
- It accumulates the component of $\mathbf{F}$ tangential to $\mathbf{r}$

Interpretation of the integral on R:

- $\int\int_R \mathrm{Curl}(\mathbf{F})\, \mathrm{d}A$ accumulates rotation of $\mathbf{F}$ in R

Interpretation of the identity: Some cancellations occur
$\hookrightarrow$ the surface integral is reduced to a curve integral

Applying Green's theorem (1)

Vector field:

$$\mathbf{F} = \left\langle y + 2, x^2 + 1 \right\rangle$$

Curve: C defined as a counterclockwise loop by

- From $(-1, 1)$ to $(1, 1)$ along $y = x^2$
- Then from $(1, 1)$ back to $(-1, 1)$ along $y = 2 - x^2$

Problem: Find

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

by two means:

1. Line integral
2. Green's theorem

Then compare both results

Applying Green's theorem (2)

Line parametrization: We have $C = C_1 \cup C_2$ with

$$\begin{aligned} C_1 : \ \mathbf{r}_1(t) &= \left\langle t,\ t^2 \right\rangle, \qquad t \text{ from } -1 \text{ to } 1 \\ C_2 : \ \mathbf{r}_2(t) &= \left\langle t,\ 2 - t^2 \right\rangle, \quad t \text{ from } 1 \text{ to } -1 \end{aligned}$$

Applying Green's theorem (3)

Line integral along C_1: We have

$$\int_{C_1} \mathbf{F} \cdot \mathbf{r}_1' \, \mathrm{d}t = \int_{-1}^{1} \left\langle t^2 + 2, t^2 + 1 \right\rangle \langle 1, 2t \rangle \, \mathrm{d}t = \frac{14}{3}$$

Line integral along C_2: We have

$$\int_{C_2} \mathbf{F} \cdot \mathbf{r}_2' \, \mathrm{d}t = \int_{1}^{-1} \left\langle 4 - t^2 + 2, t^2 + 1 \right\rangle \langle 1, -2t \rangle \, \mathrm{d}t = -\frac{22}{3}$$

Applying Green's theorem (4)

Total line integral: We have

$$\int_C \mathbf{F} \cdot \mathbf{r}' \, \mathrm{d}t = \int_{C_1} \mathbf{F} \cdot \mathbf{r}_1' \, \mathrm{d}t + \int_{C_2} \mathbf{F} \cdot \mathbf{r}_2' \, \mathrm{d}t$$
$$= \frac{14}{3} - \frac{22}{3}$$

Thus

$$\int_C \mathbf{F} \cdot \mathbf{r}' \, \mathrm{d}t = -\frac{8}{3}$$

Applying Green's theorem (5)

Rhs of Green's theorem: If $F = \langle f, g \rangle$, the rhs of (1) is

$$\int\int_R (g_x - f_y)\, \mathrm{d}A$$

Application for our F: We have

$$\begin{aligned} \mathbf{F} &= \left\langle y + 2, x^2 + 1 \right\rangle \\ g_x - f_y &= 2x - 1 \end{aligned}$$

Applying Green's theorem (6)

Area integral: We compute

$$\begin{aligned}
\int\int_R (g_x - f_y)\, \mathrm{d}A &= \int_{-1}^{1}\int_{x}^{2-x^2} (2x-1)\,\mathrm{d}y\mathrm{d}x \\
&= \int_{-1}^{1} 2xy - y\Big|_{y=x^2}^{y=2-x^2} \mathrm{d}x \\
&= 2\int_{-1}^{1} (2x-1)\left(1-x^2\right)\mathrm{d}x \\
&= -\frac{8}{3}
\end{aligned}$$

Thus

$$\int\int_R (g_x - f_y)\, \mathrm{d}A = -\frac{8}{3}$$

Applying Green's theorem (7)

Verifying Greene's theorem: We have found

$$\int_C \mathbf{F} \cdot \mathbf{r}' \, \mathrm{d}t = \int \int_R (g_x - f_y) \, \mathrm{d}A = -\frac{8}{3}$$

Green theorem and area

Theorem 14.

Let

- C simple closed curve, counterclockwise
- C delimits a region R

Then we have

$$\text{Area}(R) = \frac{1}{2} \oint_C x \, \mathrm{d}y - y \, \mathrm{d}x \qquad (2)$$

Example of area computation (1)

Curve: C defined as a counterclockwise loop by

- From $(-1, 1)$ to $(1, 1)$ along $y = x^2$
- Then from $(1, 1)$ back to $(-1, 1)$ along $y = 2 - x^2$

Problem:

Find the area for the region enclosed by the curve

Example of area computation (2)

Applying Theorem 14: We get

$$\begin{aligned}
\text{Area}(R) &= \frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x \\
&= \frac{1}{2}\left(\int_{-1}^{1} t\,(2t)\,\mathrm{d}t - \int_{-1}^{1} t^2\,\mathrm{d}t\right) \\
&\quad + \frac{1}{2}\left(\int_{1}^{-1} t\,(-2t)\,\mathrm{d}t - \int_{-1}^{1} \left(2 - t^2\right)\,\mathrm{d}t\right)
\end{aligned}$$

Thus we find

$$\text{Area}(R) = \frac{8}{3}$$

Example of area computation (3)

Usual way to compute the area:

$$\begin{aligned}\text{Area}(R) &= \int_{-1}^{1} \left[\left(2 - x^2\right) - x^2\right] \mathrm{d}x \\ &= 2\int_{-1}^{1} \left(1 - x^2\right) \mathrm{d}x\end{aligned}$$

Thus we find

$$\text{Area}(R) = \frac{8}{3}$$

Outline

Aim

Main objective for remainder of the chapter:

- Extend Green's theorem to $d = 3$

Tools:

- Notion of divergence
- Surface integral

Divergence

Definition 15.

Consider a vector field in $\mathbb{R}^3$:

$$\mathbf{F} = \langle f, g, h \rangle$$

Then the divergence of $\mathbf{F}$ is

$$\mathrm{Div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = f_x + g_y + h_z$$

Remark: In Definition 15 we have used the notation

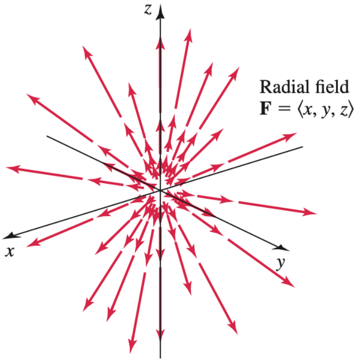
$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

Divergence for a radial field (1)

Expression for the field: We consider

$$\mathbf{F} = \langle x, y, z \rangle$$

Flux for this field:
Looking outward



Divergence for a radial field (2)

Computation of the divergence: We have

$$\nabla \cdot \mathbf{F} = f_x + g_y + h_z = 3$$

Conclusion: In this case

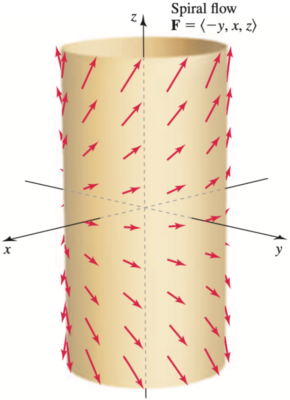
$$\text{Positive divergence} \quad \Longrightarrow \quad \text{Outward flux}$$

Divergence for a spiral field (1)

Expression for the field: We consider

$$\mathbf{F} = \langle -y, x, z \rangle$$

Flux for this field:
Spiraling upward



Divergence for a spiral field (2)

Computation of the divergence: We have

$$\nabla \cdot \mathbf{F} = f_x + g_y + h_z = 1$$

Conclusion: In this case

- Rotational part of the field does not contribute to divergence
- There is an upward flux in the z direction
- We get a positive divergence

Divergence of radial fields

Theorem 16.

Consider the vector field in $\mathbb{R}^3$ defined, for $p > 0$, by:

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p}$$

Then the divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \frac{3 - p}{|\mathbf{r}|^p}$$

Proof for $p = 1$ (1)

Expression for F: We have, if $p = 1$

$$\mathbf{F} = \frac{\langle x, y, x \rangle}{\left(x^2 + y^2 + z^2\right)^{1/2}}$$

Partial derivative: We compute f_x, that is

$$\begin{aligned}
\frac{\partial}{\partial x} \frac{x}{\left(x^2 + y^2 + z^2\right)^{1/2}} &= \frac{\left(x^2 + y^2 + z^2\right)^{1/2} - x^2 \left(x^2 + y^2 + z^2\right)^{-1/2}}{x^2 + y^2 + z^2} \\
&= \frac{|\mathbf{r}| - x^2 |\mathbf{r}|^{-1}}{|\mathbf{r}|^2} \\
&= \frac{|\mathbf{r}|^2 - x^2}{|\mathbf{r}|^3}
\end{aligned}$$

Proof for $p = 1$ (2)

Expression for the divergence: Summing the partial derivatives we get

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{3|\mathbf{r}|^2 - x^2 - z^2 - z^2}{|\mathbf{r}|^3} \\ &= \frac{2|\mathbf{r}|^2}{|\mathbf{r}|^3}\end{aligned}$$

We have found, for $p = 1$,

$$\nabla \cdot \mathbf{F} = \frac{2|\mathbf{r}|^2}{|\mathbf{r}|^3}$$

Curl

Definition 17.

Consider a vector field in $\mathbb{R}^3$:

$$\mathbf{F} = \langle f, g, h \rangle$$

Then the curl of $\mathbf{F}$ is

$$\text{Curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

Curl for a rotation field (1)

Definition of **F**: We set

$$\mathbf{F} = \mathbf{a} \times \mathbf{r}, \quad \text{with} \quad \mathbf{a} = \langle 2, -1, 1 \rangle$$

Remark:

F represents a rotation with axis *a*

Problem: Compute

Curl(**F**)

Curl for a rotation field (2)

Expression for **F**: We have

$$\begin{aligned} \mathbf{F} &= \mathbf{a} \times \mathbf{r} \\ &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & -1 & 1 \\ x & y & z \end{vmatrix} \\ &= \langle -y - z, x - 2z, x + 2y \rangle \end{aligned}$$

Curl for a rotation field (3)

Expression for Curl: We have

$$\text{Curl}(\mathbf{F}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y-z & x-2z & x+2y \end{vmatrix} = \langle 4, -2, 2 \rangle$$

Conclusion: We have found

$$\text{Curl}(\mathbf{F}) = 2\mathbf{a}$$

Summary of properties for conservative v.f

Theorem 18.

Consider a conservative vector field $\mathbf{F}$ in $\mathbb{R}^3$. Then we have

1. There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$
2. $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \varphi(B) - \varphi(A)$ for C going from A to B
3. $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ for a smooth closed curve C
4. $\mathrm{Curl}(\mathbf{F}) = 0$ at all point in $\mathbb{R}^3$

Outline

Outline

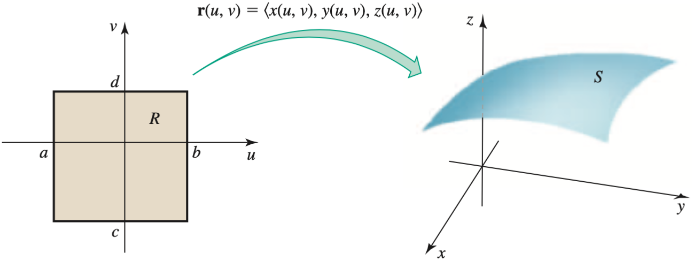
Parametrization of surfaces

Recall: Parametization of a curve in $\mathbb{R}^3$

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle , \quad a \leq t \leq b$$

Parametrization of a surface in $\mathbb{R}^3$:

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle , \quad a \leq u \leq b, \; c \leq v \leq d$$



Parametrization of a plane (1)

Surface at stake: We consider

$$S = \text{ Plane } 3x + 2y + z = 6 \quad \bigcap \quad \text{First octant}$$

Problem:

$$\text{Parametrize } S$$

Parametrization of a plane (2)

First possible parametrization: We set

$$u = x, \qquad v = y$$

Expression for z:

$$z = 6 - 3u - 2v$$

Constraints on u, v: In the xy-plane, region delimited by

$$3u + 2v = 6, \quad \text{and} \quad \text{First quadrant}$$

We get

$$\left\{0 \le u \le 2,\ 0 \le v \le 3 - \frac{3}{2}u\right\}$$

Parametrization of a plane (3)

Parametrization of the surface:

$$\mathbf{r}(u,v) = \langle u,\ v,\ 6 - 3u - 2v \rangle\,,$$

with

$$0 \le u \le 2, \quad 0 \le v \le 3 - \frac{3}{2}u$$

Parametrization of a plane (4)

Another parametrization: We set

$$u = y, \qquad v = z$$

Parametrization of the surface:

$$\mathbf{r}(u, v) = \left\langle 2 - \frac{2}{3}u - \frac{1}{3}v,\ u,\ v \right\rangle,$$

with

$$0 \le u \le 3, \quad 0 \le v \le 6 - 2u$$

Conclusion: Parametrization is not unique

Parametrization of a sphere (1)

Surface at stake: We consider

$$S = \text{ Sphere } x^2 + y^2 + z^2 = 9 \bigcap \text{ First octant } \bigcap \{1 \leq z \leq 3\}$$

Problem:

Parametrize S

Parametrization of a sphere (2)

First possible parametrization: We set

$$u = x, \qquad v = y$$

Expression for z:

$$z = \left(9 - u^2 - v^2\right)^{1/2}$$

Constraints on u, v: In the xy-plane, region delimited by

$$u^2 + v^2 = 8, \quad \text{and} \quad \text{First quadrant}$$

We get

$$\left\{0 \le u \le \sqrt{8},\ 0 \le v \le \sqrt{8 - u^2}\right\}$$

Parametrization of a sphere (3)

Parametrization of the surface:

$$\mathbf{r}(u,v) = \left\langle u,\ v,\ \left(9 - u^2 - v^2\right)^{1/2} \right\rangle,$$

with

$$0 \le u \le \sqrt{8}, \quad 0 \le v \le \sqrt{8 - u^2}$$

Parametrization of a sphere (4)

Second parametrization:
We use cylindrical coordinates r, θ, z and set

$$u = r, \quad v = \theta$$

Expression for z:

$$z = \left(9 - x^2 - y^2\right)^{1/2} = \left(9 - r^2\right)^{1/2}$$

Constraints on u, v: We get

$$\left\{0 \le u \le \sqrt{8},\ 0 \le v \le \frac{\pi}{2}\right\}$$

Parametrization of a sphere (5)

Parametrization of the surface:

$$\mathbf{r}(u,v) = \left\langle u\cos(v),\, u\sin(v),\, \left(9-u^2\right)^{1/2} \right\rangle,$$

with

$$0 \le u \le \sqrt{8}, \quad 0 \le v \le \frac{\pi}{2}$$

Parametrization of a sphere (6)

Third parametrization:
We use spherical coordinates ρ, φ, θ. Since $\rho = 3$, we set

$$u = \theta, \quad v = \varphi$$

Expression for z:

$$z = 3\cos(v)$$

Constraints on u, v: We get

$$\left\{0 \leq u \leq \frac{\pi}{2},\ 0 \leq v \leq \cos^{-1}(1/3)\right\}$$

Parametrization of a sphere (7)

Parametrization of the surface:

$$\mathbf{r}(u, v) = \langle 3\sin(v)\cos(u),\, 3\sin(v)\sin(u),\, 3\cos(v)\rangle\,,$$

with

$$0 \leq u \leq \frac{\pi}{2}, \quad 0 \leq v \leq \cos^{-1}(1/3)$$

Parametrization of a cylinder (1)

Surface at stake: We consider

$$S = \text{ Cylinder } y^2 + z^2 = 16 \bigcap \{1 \le x \le 5\}$$

Problem:

Parametrize S

Parametrization of a cylinder (2)

Possible parametrization:

Since S is a cylinder, use cylindrical yz-coordinates

$$y = r\cos(\theta), \qquad z = r\sin(\theta), \quad \text{with} \quad r = 4$$

Constraint on θ:

$$0 \le \theta \le 2\pi$$

Constraints on x:

$$1 \le x \le 5$$

Parametrization of a cylinder (3)

Parametrization of the surface:

$$\mathbf{r}(u, v) = \langle v,\ 4\cos(u),\ 4\sin(u)\rangle\ ,$$

with

$$0 \leq u \leq 2\pi, \quad 0 \leq v \leq 5$$

Outline

Approximation procedure for surface integrals

Notation: We consider

- Surface $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$
- Parameters in $R = [a, b] \times [c, d]$
- Partition of R into small rectangles R_k with left corner (u_k, v_k)
- Area of the small element of surface: ΔS_k
- Function f defined on $\mathbb{R}^3$

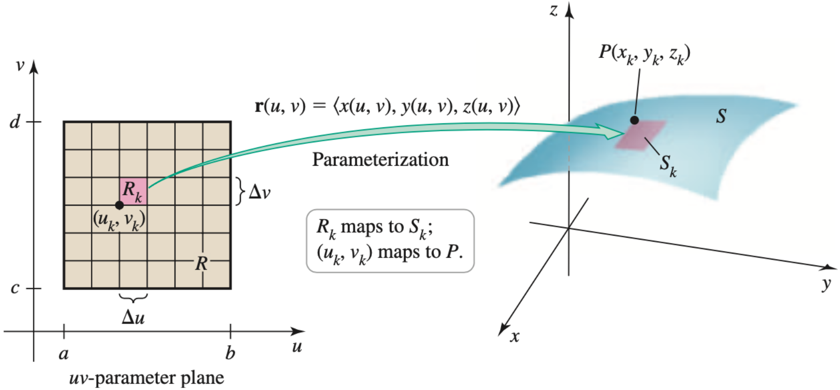
Approximation:

$$S_n = \sum_{k=1}^{n} f\left(x(u_k, v_k), y(u_k, v_k), z(u_k, v_k)\right) \Delta S_k$$

Approximation procedure: illustration

Recall:

$$S_n = \sum_{k=1}^{n} f\left(x(u_k, v_k), y(u_k, v_k), z(u_k, v_k)\right) \Delta S_k$$



Computation of ΔS_k

Tangent vectors: The tangent plane to S is generated by

$$\mathbf{t}_u = \mathbf{r}_u = \langle x_u, y_u, z_u \rangle , \quad \text{and} \quad \mathbf{t}_v = \mathbf{r}_v = \langle x_v, y_v, z_v \rangle$$

Recall: Area of parallelogram delimited by $\mathbf{w}_1, \mathbf{w}_2$ is

$$|\mathbf{w}_1 \times \mathbf{w}_2|$$

Computation of ΔS_k: We get

$$\Delta S_k \simeq |\mathbf{t}_u \times \mathbf{t}_v|$$

Computation of surface integrals in $\mathbb{R}^3$

Theorem 19.

We consider

- Surface $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$
- Parameters in $R = [a,b] \times [c,d]$
- Surface element S for $\mathbf{r}$
- Function f defined on $\mathbb{R}^3$

Then we have

$$\int_S f \, \mathrm{d}S = \int_R f\left(x(u,v), y(u,v), z(u,v)\right) |\mathbf{t}_u \times \mathbf{t}_v| \, \mathrm{d}A$$

Computation of surface integrals

Recipe:

1. Find parametric description of S
$\hookrightarrow$ $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ for $(u,v) \in [a,b] \times [c,d]$

2. Compute $|\mathbf{t}_u \times \mathbf{t}_v|$

3. Make substitutions for x and y and evaluate double integral

$$\int_R f\left(x(u,v), y(u,v), z(u,v)\right) |\mathbf{t}_u \times \mathbf{t}_v| \, \mathrm{d}A$$

Surface area of partial cylinder (1)

Surface S at stake: Cylinder

$$\{(r, \theta);\ r = 4,\ 0 \le \theta \le 2\pi\}$$

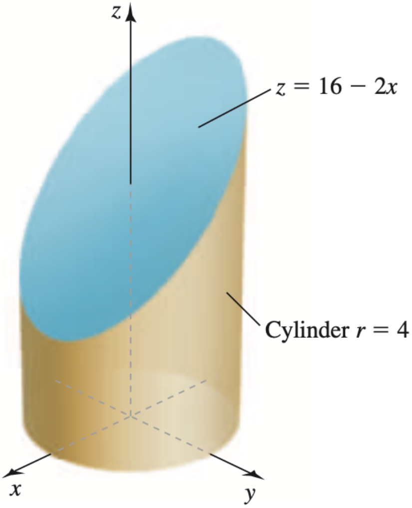
between planes

$$z = 0, \quad \text{and} \quad z = 16 - 2x$$

Problem:

Find the area of S

Surface area of partial cylinder (2)



Surface area of partial cylinder (3)

Description of the cylinder:

$$\mathbf{r}(u, v) = \langle 4\cos(u),\, 4\sin(u),\, v\rangle$$

Relation between u and v: On the plane

$$z = 16 - 2x$$

we have

$$v = 16 - 8\cos(u)$$

Region R:

$$R = \{0 \le u \le 2\pi,\, 0 \le v \le 16 - 8\cos(u)\}$$

Surface area of partial cylinder (4)

Surface element: We have

$$\begin{aligned} \mathbf{t}_u \times \mathbf{t}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4\sin(u) & 4\cos(u) & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \langle 4\cos(u), 4\sin(u), 0 \rangle \end{aligned}$$

Thus

$$|\mathbf{t}_u \times \mathbf{t}_v| = 4$$

Surface area of partial cylinder (5)

Computation of the surface area:

$$\begin{aligned}\int_S \mathbf{1}\,\mathrm{d}S &= \int_R |\mathbf{t}_u \times \mathbf{t}_v|\,\mathrm{d}A \\ &= \int_0^{2\pi}\int_0^{16-8\cos(u)} 4\,\mathrm{d}v\mathrm{d}u \\ &= 4\left(16u - 8\sin(u)\right)\Big|_0^{2\pi}\end{aligned}$$

We get

$$\int_S \mathbf{1}\,\mathrm{d}S = 128\pi$$

Average temperature on a sphere (1)

Surface S at stake: Sphere

$$\{(\rho, \varphi, \theta);\ \rho = 4,\ 0 \leq \varphi \leq \pi,\ 0 \leq \theta \leq 2\pi,\}$$

Temperature distribution: Cooler at the poles, warmer at the equator,

$$T(\varphi, \theta) = 10 + 50 \sin(\varphi)$$

Problem:

Find the average temperature on S

Average temperature on a sphere (2)

Description of the sphere:

$$\mathbf{r}(u, v) = \langle 4\sin(u)\cos(v),\ 4\sin(u)\sin(v), 4\cos(u)\rangle$$

Expression for the temperature: In terms of u, v,

$$T = f(u, v) = 10 + 50\sin(u)$$

Region R:

$$R = \{0 \le u \le \pi,\ 0 \le v \le 2\pi\}$$

Average temperature on a sphere (3)

Surface element: We have

$$\begin{aligned}
\mathbf{t}_u \times \mathbf{t}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4\cos(u)\cos(v) & 4\cos(u)\sin(v) & -4\sin(u) \\ -4\sin(u)\sin(v) & 4\sin(u)\cos(v) & 0 \end{vmatrix} \\
&= 16\left\langle \sin^2(u)\cos(v), \sin^2(u)\sin(v), \sin(u)\cos(u) \right\rangle
\end{aligned}$$

Thus

$$|\mathbf{t}_u \times \mathbf{t}_v| = 16\sin(u)$$

Average temperature on a sphere (4)

Computation of the average temperature:

$$\begin{aligned}\int_S (10+50\sin(u))\,\mathrm{d}S &= \int_R (10+50\sin(u))\,|\mathbf{t}_u\times\mathbf{t}_v|\,\mathrm{d}A\\ &= \int_0^{\pi}\int_0^{2\pi}(10+50\sin(u))\cdot 16\sin(u)\,\mathrm{d}v\mathrm{d}u\\ &= 32\pi\int_0^{\pi}(10+50\sin(u))\sin(u)\,\mathrm{d}u\\ &= 160\pi(4+5\pi)\end{aligned}$$

We get

$$\bar{T} = \frac{160\pi(4+5\pi)}{4\pi\cdot 16} = \frac{20+25\pi}{2} \simeq 49.3$$

Surface integrals in $\mathbb{R}^3$ in the explicit case

Theorem 20.

We consider

- Surface S explicitely given by $z = g(x, y)$
- Parameters in $(x, y) \in R$
- Function f defined on $\mathbb{R}^3$

Then we have

$$\int_S f \, \mathrm{d}S = \int_R f\left(x, y, g(x, y)\right) \sqrt{z_x^2 + z_y^2 + 1} \, \mathrm{d}A$$

Surface area of a roof (1)

Surface S at stake: In the plane

$$z = 12 - 4x - 3y\,,$$

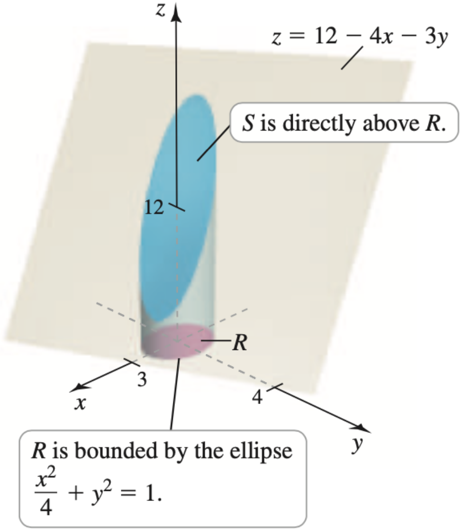
directly above the region R bounded by ellipse

$$\frac{x^2}{4} + y^2 = 1$$

Problem:

Find the area of S

Surface area of a roof (2)



Surface area of a roof (3)

Region R:

$$\frac{x^2}{4} + y^2 = 1$$

Function f: Since we just compute an area, we take

$$f = \mathbf{1}$$

Surface element: We have

$$z_x = -4, \qquad z_y = -3,$$

thus

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{26}$$

Surface area of a roof (4)

Computation of the surface area:

$$
\begin{aligned}
\int_S \mathbf{1}\, \mathrm{d}S &= \int_R \sqrt{z_x^2 + z_y^2 + 1}\, \mathrm{d}A \\
&= \sqrt{26} \int_R \mathrm{d}A \\
&= \sqrt{26}\, \mathrm{Area}(R)
\end{aligned}
$$

We get (area of an ellipse is $\pi\, a\, b$)

$$
\int_S \mathbf{1}\, \mathrm{d}S = 2\pi\sqrt{26}
$$

Mass of a conical sheet (1)

Surface S at stake: Cone of the form

$$z = \left(x^2 + y^2\right)^{1/2},$$

together with the constraint

$$0 \leq z \leq 4$$

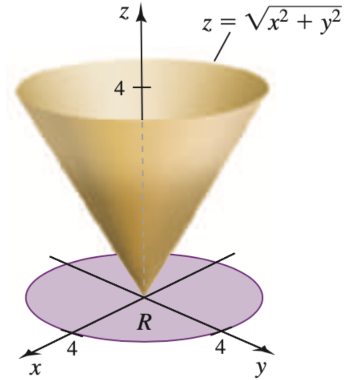
Mass density: Heavier close to the bottom, of the form

$$f(x, y, z) = 8 - z$$

Problem:

Find the total mass of S

Mass of a conical sheet (2)



Mass of a conical sheet (3)

Region R: Corresponding to $z \leq 4$ we get

$$R = \left\{ x^2 + y^2 \leq 4 \right\}$$

Function f: The mass density is

$$f = 8 - z = 8 - \left(x^2 + y^2 \right)^{1/2}$$

Surface element: We have $z_x = \frac{x}{z}$ and $z_y = \frac{x}{z}$, thus

$$\sqrt{z_x^2 + z_y^2 + 1} = \left((x/z)^2 + (y/z)^2 + 1 \right)^{1/2} = \sqrt{2}$$

Mass of a conical sheet (4)

Computation of the surface area:

$$\begin{aligned}
\int_S f(x,y,z)\,\mathrm{d}S &= \int_R f(x,y,z)\sqrt{z_x^2+z_y^2+1}\,\mathrm{d}A \\
&= \sqrt{2}\int_R \left(8-\left(x^2+y^2\right)^{1/2}\right)\,\mathrm{d}A \\
&= \sqrt{2}\int_0^{2\pi}\int_0^4 (8-r)r\,\mathrm{d}r\mathrm{d}\theta \\
&= \sqrt{2}\int_0^{2\pi}\left(4r^2-\frac{r^3}{3}\right)\Big|_0^4\,\mathrm{d}\theta
\end{aligned}$$

We get

$$\int_S f(x,y,z)\,\mathrm{d}S = \frac{256\pi\sqrt{2}}{3} \simeq 379$$

Summary of descriptions for common surfaces

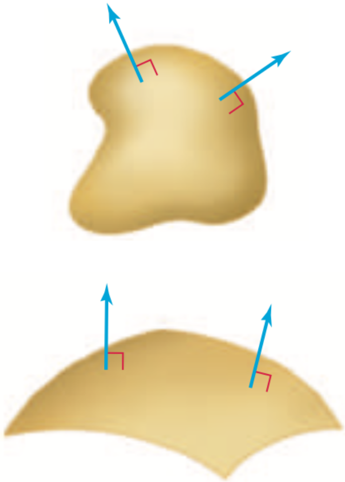
	Explicit Description $z = g(x,y)$		Parametric Description	
Surface	**Equation**	**Normal vector; magnitude** $\pm\langle -z_x, -z_y, 1\rangle; \lvert\langle -z_x, -z_y, 1\rangle\rvert$	**Equation**	**Normal vector; magnitude** $t_u \times t_v; \lvert t_u \times t_v\rvert$
Cylinder	$x^2 + y^2 = a^2$, $0 \le z \le h$	$\langle x, y, 0\rangle; a$	$\mathbf{r} = \langle a\cos u, a\sin u, v\rangle$, $0 \le u \le 2\pi, 0 \le v \le h$	$\langle a\cos u, a\sin u, 0\rangle; a$
Cone	$z^2 = x^2 + y^2$, $0 \le z \le h$	$\langle x/z, y/z, -1\rangle; \sqrt{2}$	$\mathbf{r} = \langle v\cos u, v\sin u, v\rangle$, $0 \le u \le 2\pi, 0 \le v \le h$	$\langle v\cos u, v\sin u, -v\rangle; \sqrt{2}v$
Sphere	$x^2 + y^2 + z^2 = a^2$	$\langle x/z, y/z, 1\rangle; a/z$	$\mathbf{r} = \langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle$, $0 \le u \le \pi, 0 \le v \le 2\pi$	$\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\sin u\cos u\rangle; a^2\sin u$
Paraboloid	$z = x^2 + y^2$, $0 \le z \le h$	$\langle 2x, 2y, -1\rangle; \sqrt{1 + 4(x^2 + y^2)}$	$\mathbf{r} = \langle v\cos u, v\sin u, v^2\rangle$, $0 \le u \le 2\pi, 0 \le v \le \sqrt{h}$	$\langle 2v^2\cos u, 2v^2\sin u, -v\rangle; v\sqrt{1 + 4v^2}$

Outline

Surface orientation (1)

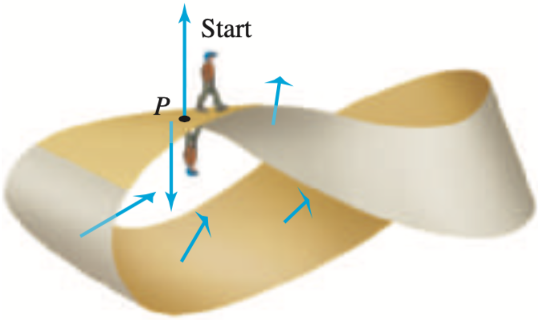
Basic principle of orientation:

Normal vectors point in the outward direction



Surface orientation (2)

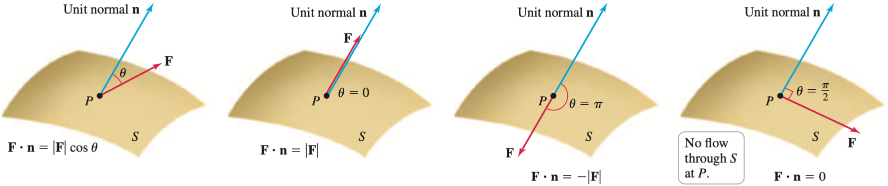
Warning: Not every surface admits an orientation!



Flux

Common situation:

- We have a vector field **F** in $\mathbb{R}^3$
- **F** represents the flow of a fluid
- We wish to compute the flow of **F** across a surface S
- This is given by a surface integral



Surface integral of a vector field, parametric case

Definition 21.

Consider

- Vector field $\mathbf{F} = \langle f, g, h \rangle$ in $\mathbb{R}^3$
- Surface S defined for $(u, v) \in R$ by

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

- $\mathbf{t}_u$, $\mathbf{t}_v$ tangent vectors for S
  $\hookrightarrow$ With $\mathbf{t}_u \times \mathbf{t}_v$ respecting the orientation of S

Then we set

$$\int\int_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \int\int_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) \, \mathrm{d}A$$

Example of surface integral (1)

Vector field: We consider

$$\mathbf{F} = \langle x, y, z \rangle$$

Surface: Plane

$$S: \quad 3x + 2y + z = 6 \quad \bigcap \quad \text{First octant},$$

with normal vector pointing upward

Problem: Compute

$$\int\int_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S$$

Example of surface integral (2)

Parametrization of S: We take

$$\mathbf{r}(u,v) = \langle u, v, 6-3u-2v \rangle, \quad (u,v) \in R,$$

with

$$R = \left\{ 0 \le u \le 2,\ 0 \le v \le 3 - \frac{3}{2}u \right\}$$

Normal vector: We have

$$\mathbf{t}_u = \langle 1, 0, -3 \rangle, \quad \mathbf{t}_v = \langle 0, 1, -2 \rangle, \quad \mathbf{t}_u \times \mathbf{t}_v = \langle 3, 2, 1 \rangle$$

Note:
$\mathbf{t}_u \times \mathbf{t}_v$ is conveniently oriented upward (positive z-component)

Example of surface integral (3)

Surface integral: We get

$$\begin{aligned}\int\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S &= \int\int_R \mathbf{F}\cdot(\mathbf{t}_u\times\mathbf{t}_v)\,\mathrm{d}A \\ &= \int_0^2\int_0^{3-\frac{3}{2}u}\langle u, v, 6-3u-2v\rangle\cdot\langle 3,2,1\rangle\,\mathrm{d}u\mathrm{d}v \\ &= 6\int_0^2\int_0^{3-\frac{3}{2}u}\mathrm{d}u\mathrm{d}v\end{aligned}$$

We get

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = 18$$

Surface integral on a sphere (1)

Vector field: We consider

$$\mathbf{F} = -\frac{\langle x, y, z\rangle}{(x^2 + y^2 + z^2)^{3/2}}$$

Surface: Plane

$$S: \quad \text{Sphere of radius } a\text{, normal outward,}$$

Problem: Compute

$$\int\int_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S$$

Surface integral on a sphere (2)

Parametrization of S: We take

$$\mathbf{r}(u,v) = \langle a\sin(u)\cos(v),\ a\sin(u)\sin(v), a\cos(u)\rangle\,, \quad (u,v) \in R,$$

with

$$R = \{0 \le u \le \pi,\, 0 \le v \le 2\pi\}$$

Surface integral on a sphere (3)

Normal vector: We have

$$\begin{aligned} \mathbf{t}_u &= \langle a\cos(u)\cos(v),\ a\cos(u)\sin(v), -a\sin(u)\rangle\,, \\ \mathbf{t}_v &= \langle -a\sin(u)\sin(v),\ a\sin(u)\cos(v), 0\rangle\,, \end{aligned}$$

Thus

$$\mathbf{t}_u \times \mathbf{t}_v = \left\langle a^2\sin^2(u)\cos(v),\ a^2\sin^2(u)\sin(v), a^2\cos(u)\sin(u)\right\rangle$$

Note: $\mathbf{t}_u \times \mathbf{t}_v$ is conveniently oriented outward
$\hookrightarrow$ Example: for $u = \frac{\pi}{2}$ and $v = 0$ we have

$$\mathbf{r}(u,v) = \langle a, 0, 0\rangle\,, \quad \mathbf{t}_u \times \mathbf{t}_v = \left\langle a^2, 0, 0\right\rangle$$

Surface integral on a sphere (4)

Surface integral: We get

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \int\int_R \mathbf{F}\cdot(\mathbf{t}_u\times\mathbf{t}_v)\,\mathrm{d}A$$

$$= -\int_0^{2\pi}\int_0^{\pi} \frac{\langle a\sin(u)\cos(v),\ a\sin(u)\sin(v), a\cos(u)\rangle}{a^3}$$

$$\cdot\left\langle a^2\sin^2(u)\cos(v),\ a^2\sin^2(u)\sin(v), a^2\cos(u)\sin(u)\right\rangle\,\mathrm{d}u\mathrm{d}v$$

$$= -\int_0^{2\pi}\int_0^{\pi}\left(\sin^3(u)\cos^2(v)+\sin^3(u)\sin^2(v)+\cos^2(u)\sin(u)\right)\mathrm{d}u\mathrm{d}v$$

$$= -\int_0^{2\pi}\int_0^{\pi}\sin(u)\,\mathrm{d}u\mathrm{d}v$$

We get a negative flux (since **F** points inward):

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = -4\pi$$

Surface integral of a vector field, explicit case

Definition 22.

Consider

- Vector field $\mathbf{F} = \langle f, g, h \rangle$ in $\mathbb{R}^3$
- Surface S defined for $(x, y) \in R$ by

$$z = s(x, y)$$

Then we have

$$\int\int_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \int\int_R \left(-f \, z_x - g \, z_y + h \right) \mathrm{d}A$$

Outline

The main theorem

Theorem 23.

Consider

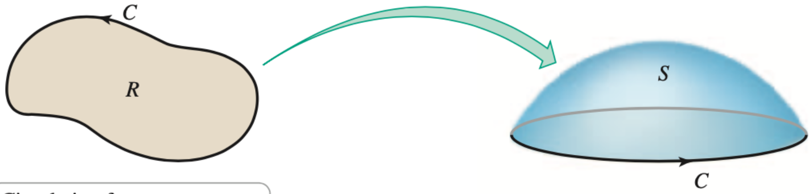
- An oriented surface S in $\mathbb{R}^3$
- S has a smooth boundary C
- $\mathbf{F} = \langle f, g, h \rangle$ vector field in $\mathbb{R}^3$
- $\mathrm{Curl}(\mathbf{F}) = \nabla \times \mathbf{F}$

Then we have

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S$$

From Green to Stokes

From 2-d to 3-d:



Circulation form of Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$$

Stokes' Theorem:

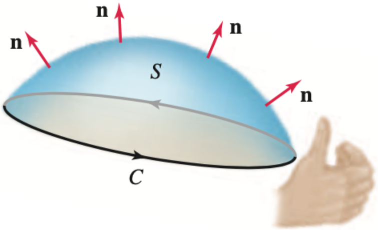
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

Orientations

Compatibility of orientations: Stokes' theorem involves

- An oriented surface
- An oriented curve (counterclockwise)

The orientations have to be compatible through the right hand rule



Verifying Stokes theorem (1)

Vector field:

$$\mathbf{F} = \langle z - y, x, -x \rangle$$

Surface: Hemisphere

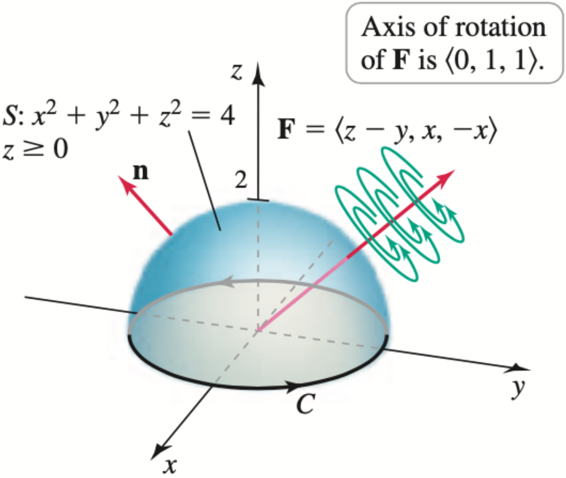
$$S : \quad x^2 + y^2 + z^2 = 4 \quad \bigcap \quad \{z \geq 0\}$$

Corresponding curve: In xy-plane, circle oriented counterclockwise

$$C : \quad x^2 + y^2 = 4$$

Problem:
Verify Stokes' theorem in this context

Verifying Stokes theorem (2)



Verifying Stokes theorem (3)

Parametric equation for C:

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 0\rangle$$

Parametric equation for $\mathbf{F}$: Along C we have

$$\mathbf{F} = \langle z - y, x, -x\rangle = 2\,\langle -\sin(t), \cos(t), -\cos(t)\rangle$$

Dot product: We have

$$\mathbf{F}(t)\cdot\mathbf{r}'(t) = \left(\cos^2(t) + \sin^2(t)\right) = 4$$

Verifying Stokes theorem (4)

Line integral:

$$\begin{aligned}\oint_C \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s &= \oint_C \mathbf{F}(t) \cdot \mathbf{r}'(t) \, \mathrm{d}t \\ &= 4 \int_0^{2\pi} \mathrm{d}t\end{aligned}$$

Thus we get

$$\oint_C \mathbf{F}(t) \cdot \mathbf{r}'(t) \, \mathrm{d}t = 8\pi$$

Verifying Stokes theorem (5)

Expression for $\mathrm{Curl}(\mathbf{F})$: We have

$$\mathrm{Curl}(\mathbf{F}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & x & -x \end{vmatrix}$$

Computation: We find that $\mathbf{F}$ is a rotation with axis $\langle 0, 1, 1 \rangle$

$$\mathrm{Curl}(\mathbf{F}) = \langle 0, 2, 2 \rangle$$

Verifying Stokes theorem (6)

Parametrization of S: We take

$$\mathbf{r}(u,v) = \langle 2\sin(u)\cos(v),\, 2\sin(u)\sin(v), 2\cos(u)\rangle\,, \quad (u,v)\in R,$$

with

$$R = \{0 \le u \le \pi/2,\, 0 \le v \le 2\pi\}$$

Verifying Stokes theorem (7)

Normal vector: We have

$$\begin{aligned} \mathbf{t}_u &= \langle 2\cos(u)\cos(v),\ 2\cos(u)\sin(v), -2\sin(u)\rangle\,, \\ \mathbf{t}_v &= \langle -2\sin(u)\sin(v),\ 2\sin(u)\cos(v), 0\rangle\,, \end{aligned}$$

Thus

$$\mathbf{t}_u \times \mathbf{t}_v = \left\langle 4\sin^2(u)\cos(v),\ 4\sin^2(u)\sin(v), 4\cos(u)\sin(u)\right\rangle$$

Verifying Stokes theorem (8)

Surface integral: We get

$$\int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S = \int\int_R \mathrm{Curl}(\mathbf{F}) \cdot (\mathbf{t}_u \times \mathbf{t}_v) \, \mathrm{d}A$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \langle 0, 2, 2 \rangle \cdot \left\langle 4\sin^2(u)\cos(v), \, 4\sin^2(u)\sin(v), 4\cos(u)\sin(u) \right\rangle \, \mathrm{d}u\mathrm{d}v$$

$$= 8 \int_0^{2\pi} \int_0^{\pi/2} \left(\sin^2(u)\sin(v) + \sin(u)\cos(u) \right) \, \mathrm{d}u\mathrm{d}v$$

$$= 8 \int_0^{2\pi} \int_0^{\pi/2} sin(u)\cos(u) \, \mathrm{d}u\mathrm{d}v$$

We get a positive flux (since **F** points outward):

$$\int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S = 8\pi$$

Verifying Stokes theorem (9)

Verification: We have found

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S = 8\pi$$

Stokes theorem for a line integral (1)

Vector field:

$$\mathbf{F} = \left\langle z, -z, x^2 - y^2 \right\rangle$$

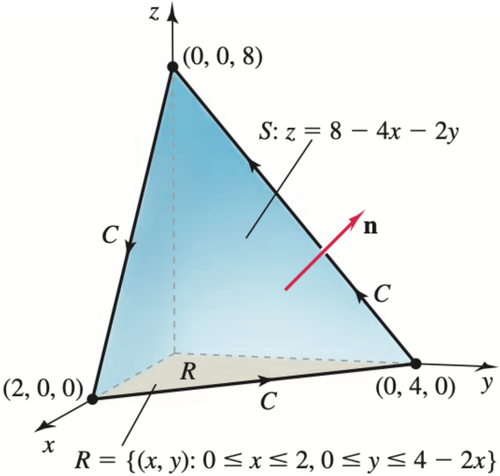
Surface: Plane in the first octant, with $\mathbf{n}$ pointing upward

$$S: \quad z = 8 - 4x - 2y \quad \bigcap \quad \{x \geq 0,\ y \geq 0,\ z \geq 0\}$$

Corresponding curve:
Three lines delimiting S

Problem: In order to avoid a parametrization of C
$\hookrightarrow$ Evaluate $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ as a surface integral

Stokes theorem for a line integral (2)



Stokes theorem for a line integral (3)

Expression for $\mathrm{Curl}(\mathbf{F})$: We have

$$\mathrm{Curl}(\mathbf{F}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -z & x^2 - y^2 \end{vmatrix}$$

Computation: We find

$$\mathrm{Curl}(\mathbf{F}) = \langle 1 - 2y,\, 1 - 2x,\, 0 \rangle$$

Stokes theorem for a line integral (4)

Parametrization of S: We take the explicit version

$$z = 8 - 4x - 2y, \quad (x, y) \in R,$$

with

$$R = \{0 \leq x \leq 2,\, 0 \leq y \leq 4 - 2x\}$$

Stokes theorem for a line integral (5)

Normal vector: We write the plane as

$$4x + 2y + z = 8$$

Thus

$$\mathbf{n} = \langle 4, 2, 1 \rangle$$

Formula used for the surface integral: Explicit case in Definition 22

$$\int\int_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \int\int_R (-f \, z_x - g \, z_y + h) \, \mathrm{d}A$$

Stokes theorem for a line integral (6)

Surface integral: We get

$$\int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S = \int_0^2 \int_0^{4-2x} \langle 4, 2, 1 \rangle \cdot \langle 1 - 2y, \, 1 - 2x, \, 0 \rangle \, \mathrm{d}x\mathrm{d}y = \int_0^2 \int_0^{4-2x} (6 - 4x - 8y) \, \mathrm{d}x\mathrm{d}y$$

We obtain:

$$\int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S = -\frac{88}{3}$$

Stokes theorem for a line integral (7)

Computation of the line integral: We have

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S = -\frac{88}{3}$$

Remark:

We get a negative flux (circulation is going clockwise)

Stokes theorem for a surface integral (1)

Vector field:

$$\mathbf{F} = \langle -y, x, z \rangle$$

Surface: Part of a paraboloid within another paraboloid

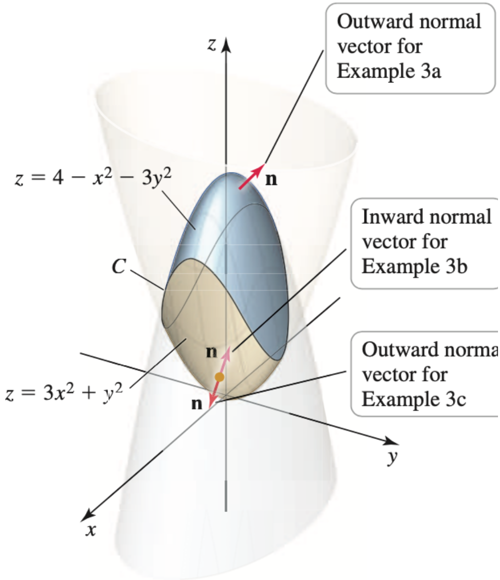
$$S: \quad z = 4 - x^2 - 3y^2 \quad \bigcap \quad \left\{ z \geq 3x^2 + y^2 \right\},$$

with $\mathbf{n}$ pointing upward

Corresponding curve:
Intersection of the 2 paraboloids

Problem: In order to avoid a parametrization of S
$\hookrightarrow$ Evaluate $\int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S$ as a line integral

Stokes theorem for a surface integral (2)



Stokes theorem for a surface integral (3)

Equation for C: For the intersection of the paraboloids we get

$$4 - x^2 - 3y^2 = 3x^2 + y^2 \quad \Longleftrightarrow \quad x^2 + y^2 = 1$$

Parametric equation for x, y: We choose

$$x = \cos(t), \quad y = \sin(t), \qquad 0 \le t \le 2\pi,$$

which is compatible with the orientation of S

Parametric equation for C: Writing $z = 3x^2 + y^2$ we get

$$\mathbf{r}(t) = \left\langle \cos(t), \sin(t), 3\cos^2(t) + \sin^2(t) \right\rangle$$

Stokes theorem for a surface integral (4)

Parametric equation for $\mathbf{F}$: Along C we have

$$\mathbf{F} = \langle -y, x, z \rangle = \left\langle -\sin(t), \cos(t), 3\cos^2(t) + \sin^2(t) \right\rangle$$

Dot product: We have

$$\mathbf{F}(t) \cdot \mathbf{r}'(t) = \left\langle -\sin(t), \cos(t), 3\cos^2(t) + \sin^2(t) \right\rangle \cdot \langle -\sin(t), \cos(t), -4\cos(t)\sin(t) \rangle$$

We get

$$\mathbf{F}(t) \cdot \mathbf{r}'(t) = 1 - 12\cos^3(t)\sin(t) - 4\sin^3(t)\cos(t)$$

Stokes theorem for a surface integral (5)

Line integral:

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s &= \oint_C \mathbf{F}(t)\cdot\mathbf{r}'(t)\,\mathrm{d}t \\ &= \int_0^{2\pi}\,\mathrm{d}t\end{aligned}$$

Thus we get

$$\oint_C \mathbf{F}(t)\cdot\mathbf{r}'(t)\,\mathrm{d}t = 2\pi$$

Stokes theorem for a surface integral (6)

Computation of the surface integral: We have

$$\int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 2\pi$$

Remark:
We get a positive flux (normal is oriented like $\mathrm{Curl}(\mathbf{F})$)

Outline

The main theorem

Theorem 24.

Consider

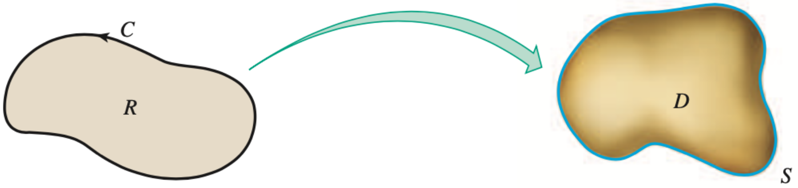
- A simply connected region D in $\mathbb{R}^3$
- D is enclosed by an oriented surface S
- $\mathbf{F} = \langle f, g, h \rangle$ vector field in $\mathbb{R}^3$
- $\mathrm{Div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$

Then we have

$$\int\int_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \int\int\int_D \mathrm{Div}(\mathbf{F}) \, \mathrm{d}V$$

From Green to divergence

From 2-d to 3-d:



Flux form of Green's Theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA$$

Divergence Theorem:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \operatorname{div} \mathbf{F}\, dV$$

Verifying divergence theorem (1)

Vector field:

$$\mathbf{F} = \langle x, y, z \rangle$$

Surface: Sphere S of the form

$$S : \quad x^2 + y^2 + z^2 = a^2$$

Corresponding domain: Ball of the form

$$B = \left\{ x^2 + y^2 + z^2 \leq a^2 \right\}$$

Problem:
Verify divergence theorem in this context

Verifying divergence theorem (2)

Expression for $\mathrm{Div}(\mathbf{F})$: We have

$$\mathrm{Div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$$

Computation: We find

$$\mathrm{Div}(\mathbf{F}) = 3$$

Verifying divergence theorem (3)

Volume integral: We have

$$\begin{aligned}\int\int\int_D \mathrm{Div}(\mathbf{F})\,\mathrm{d}V &= 3\int\int\int_D \mathrm{d}V \\ &= 3\,\mathrm{Vol}(D)\end{aligned}$$

Thus

$$\int\int\int_D \mathrm{Div}(\mathbf{F})\,\mathrm{d}V = 4\pi a^3$$

Verifying divergence theorem (4)

Parametrization of S: We take

$$\mathbf{r}(u,v) = \langle a\sin(u)\cos(v),\ a\sin(u)\sin(v), a\cos(u)\rangle\,, \quad (u,v) \in R,$$

with

$$R = \{0 \le u \le \pi,\, 0 \le v \le 2\pi\}$$

Verifying divergence theorem (5)

Normal vector: We have

$$\begin{aligned} \mathbf{t}_u &= \langle a\cos(u)\cos(v),\ a\cos(u)\sin(v), -a\sin(u)\rangle , \\ \mathbf{t}_v &= \langle -a\sin(u)\sin(v),\ a\sin(u)\cos(v), 0\rangle , \end{aligned}$$

Thus

$$\mathbf{t}_u \times \mathbf{t}_v = \left\langle a^2\sin^2(u)\cos(v),\ a^2\sin^2(u)\sin(v), a^2\cos(u)\sin(u)\right\rangle$$

Verifying divergence theorem (6)

Surface integral: We get

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \int\int_R \mathbf{F}\cdot(\mathbf{t}_u\times\mathbf{t}_v)\,\mathrm{d}A$$
$$= \int_0^{2\pi}\int_0^{\pi} \langle a\sin(u)\cos(v),\ a\sin(u)\sin(v), a\cos(u)\rangle$$
$$\cdot\left\langle a^2\sin^2(u)\cos(v),\ a^2\sin^2(u)\sin(v), a^2\cos(u)\sin(u)\right\rangle\,\mathrm{d}u\mathrm{d}v$$
$$= a^3\int_0^{2\pi}\int_0^{\pi}\sin(u)\,\mathrm{d}u\mathrm{d}v$$

We get

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = 4\pi a^3$$

Verifying divergence theorem (7)

Verification: We have found

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \int\int\int_D \mathrm{Div}(\mathbf{F})\,\mathrm{d}V = 4\pi a^3$$

Computing a flux with the divergence (1)

Vector field:

$$\mathbf{F} = xyz \langle 1, 1, 1 \rangle$$

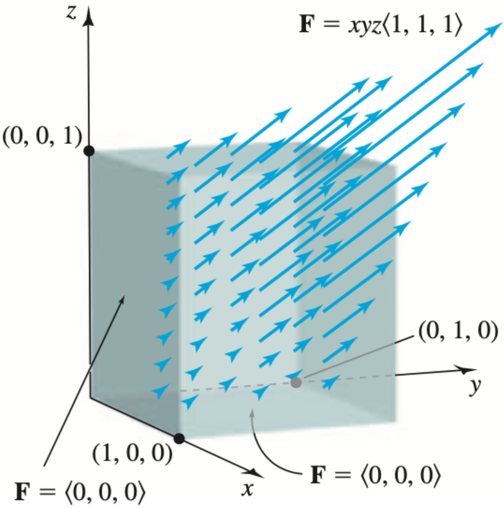
Domain: Cube of the form

$$D: \quad \{0 \le x \le 1,\, 0 \le y \le 1,\, 0 \le z \le 1\}$$

Corresponding surface S:
6 faces of the cube

Problem: In order to avoid a parametrization of S
$\hookrightarrow$ Evaluate $\int\int_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S$ as a volume integral

Computing a flux with the divergence (2)



Computing a flux with the divergence (3)

Expression for $\mathrm{Div}(\mathbf{F})$: We have

$$\mathrm{Div}(\mathbf{F}) = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(xyz)$$

Computation: We find

$$\mathrm{Div}(\mathbf{F}) = yz + xz + xy$$

Computing a flux with the divergence (4)

Volume integral: We get

$$\int\int\int_D \mathrm{Div}(\mathbf{F})\,\mathrm{d}V$$
$$= \int_0^1\int_0^1\int_0^1 (yz + xz + xy)\ \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

We obtain:

$$\int\int\int_D \mathrm{Div}(\mathbf{F})\,\mathrm{d}V = \frac{3}{4}$$

Computing a flux with the divergence (5)

Computation of the surface integral: The flux of **F** through S is

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \int\int\int_D \mathrm{Div}(\mathbf{F})\,\mathrm{d}V = \frac{3}{4}$$

Remark:
We get a positive outward flux