Vector-valued functions

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Multivariate calculus - MA 261

Mostly taken from *Calculus, Early Transcendentals*
by Briggs - Cochran - Gillett - Schulz

Outline

Outline

Functions with values in $\mathbb{R}^3$

Scalar-valued functions: We are used to functions like

$$f(t) = 3t^2 + 5 \quad \Longrightarrow \quad f(1) = 8 \in \mathbb{R}$$

Vector-valued functions: In this course we consider

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \quad \Longrightarrow \quad \mathbf{r}(t) \in \mathbb{R}^3$$

Lines as vector-valued functions (1)

Problem: Consider the line passing through

$$P(1,2,3) \quad \text{and} \quad Q(4,5,6)$$

Find a vector-valued function for this line

Lines as vector-valued functions (2)

Parallel vector:

$$\mathbf{v} = (3, 3, 3), \quad \text{simplified as} \quad \mathbf{v} = (1, 1, 1)$$

Equation for the line:

$$\mathbf{r}(t) = \langle 1 + t, 2 + t, 3 + t \rangle$$

Examples of points:

$$\mathbf{r}(0) = \langle 1, 2, 3 \rangle\,, \quad \mathbf{r}(1) = \langle 2, 3, 4 \rangle\,, \quad \mathbf{r}(2) = \langle 3, 4, 5 \rangle$$

Spiral (1)

Problem: Graph the curve defined by

$$\mathbf{r}(t) = \left\langle 4\cos(t), \sin(t), \frac{t}{2\pi} \right\rangle$$

Spiral (2)

Projection on xy-plane: Set $z = 0$. We get

$$\langle 4\cos(t), \sin(t) \rangle$$

This is an ellipse, counterclockwise, starts at $(4, 0, 0)$

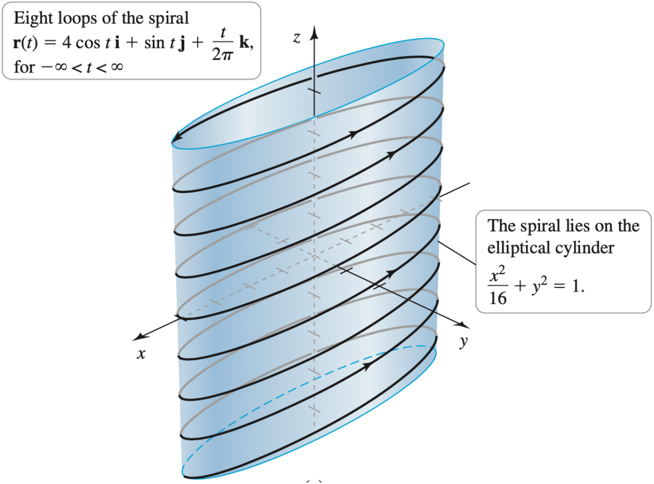
Related surface: We have

$$\frac{x^2}{4} + y^2 = 1$$

Thus curve lies on an elliptic cylinder

Upward direction: The z-component is $\frac{t}{2\pi}$
$\hookrightarrow$ Spiral on the cylinder

Spiral (3)



Eight loops of the spiral
$\mathbf{r}(t) = 4\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \dfrac{t}{2\pi}\mathbf{k}$,
for $-\infty < t < \infty$
z
x
y
The spiral lies on the elliptical cylinder
$\dfrac{x^2}{16} + y^2 = 1.$

Domain of vector-valued functions

Definition: The domain of $t \mapsto \mathbf{r}(t)$ is
$\hookrightarrow$ The intersection of the domains for each component

Example: If

$$\mathbf{r}(t) = \left\langle \sqrt{1-t^2}, \sqrt{t}, \frac{1}{\sqrt{5+t}} \right\rangle,$$

then the domain of $\mathbf{r}$ is

$$[0, 1]$$

Limits and continuity (1)

Function: We define

$$\mathbf{r}(t) = \left\langle \cos(\pi\, t),\, \sin(\pi\, t), e^{-t} \right\rangle$$

Questions:

1. Graph $\mathbf{r}$
2. Evaluate $\lim_{t\to 2} \mathbf{r}(t)$
3. Evaluate $\lim_{t\to\infty} \mathbf{r}(t)$
4. At what points is $\mathbf{r}$ continuous?

Limits and continuity (2)

Answers

1. $\lim_{t\to 2} \mathbf{r}(t) = \langle 1, 0, e^{-2} \rangle$
2. No limit. As $t \to \infty$
   $\hookrightarrow$ $\mathbf{r}(t)$ approaches the unit circle in xy-plane
3. $\mathbf{r}$ is continuous everywhere

Outline

Derivative

Definition 1.

Let

- $\mathbf{r}(t)$ a vector-valued function
- $\mathbf{r}$ of the form $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$

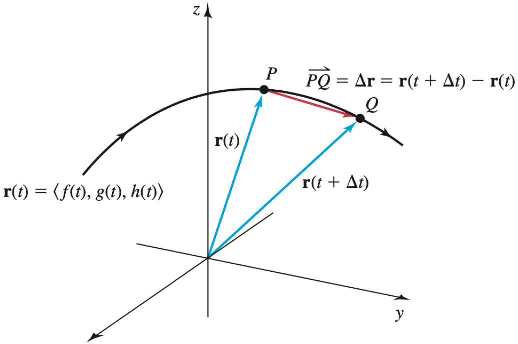
Then the derivative of $\mathbf{r}$ is defined by

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

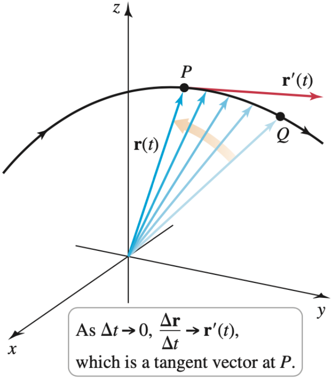
We also have

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Derivative and velocity



(a)



(b)

Spiral on cone example

Function: Consider the curve defined by

$$\mathbf{r}(t) = \langle t\cos(t), t\sin(t), t\rangle$$

Derivative: We get

$$\mathbf{r}'(t) = \langle -t\sin(t) + \cos(t), t\cos(t) + \sin(t), 1\rangle$$

Related surface: $\mathbf{r}$ is a spiral on the cone

$$x^2 + y^2 = z^2$$

Unit tangent vector

Definition 2.

Let

- $\mathbf{r}(t)$ a vector-valued function
- Assume $\mathbf{r}'(t) \neq 0$

Then the unit tangent vector of $\mathbf{r}$ at time t is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Spiral on cone example

Function: Consider the curve defined by

$$\mathbf{r}(t) = \langle t\cos(t), t\sin(t), t\rangle$$

Derivative: We have seen

$$\mathbf{r}'(t) = \langle -t\sin(t) + \cos(t), t\cos(t) + \sin(t), 1\rangle$$

Unit tangent: We get

$$\mathbf{T}(t) = \left\langle \frac{-t\sin(t) + \cos(t)}{\sqrt{t^2+2}}, \frac{t\cos(t) + \sin(t)}{\sqrt{t^2+2}}, \frac{1}{\sqrt{t^2+2}} \right\rangle$$

Product rules

Theorem 3.

Let

- $\mathbf{u}, \mathbf{v}$ vector-valued functions
- f real-valued function

Then we have

$$
\begin{aligned}
[f(t)\mathbf{u}(t)]' &= f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) \\
[\mathbf{u}(t) \cdot \mathbf{v}(t)]' &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \\
[\mathbf{u}(t) \times \mathbf{v}(t)]' &= \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)
\end{aligned}
$$

Example of product rule

Functions: Consider

$$\mathbf{r}(t) = \left\langle 1, t, t^2 \right\rangle, \qquad f(t) = e^t$$

Product derivative: We find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f(t)\mathbf{r}(t)\right] = e^t \left\langle 1, t+1, t^2+2t \right\rangle$$

Antiderivative

Definition 4.

Consider

- $\mathbf{r}$ of the form $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$
- F, G, H antiderivatives of f, g, h respectively
- $\mathbf{R}(t) = \langle F(t), G(t), H(t) \rangle$

Then we have

$$\int \mathbf{r}(t)\, \mathrm{d}t = R(t) + \langle C_1, C_2, C_3 \rangle$$

Example of antiderivative

Function: Consider

$$\mathbf{r}(t) = \left\langle \frac{t}{\sqrt{t^2+2}}, e^{-3t}, \sin(4t) + 1 \right\rangle$$

Antiderivative: We get

$$\int \mathbf{r}(t)\,\mathrm{d}t = \left\langle \sqrt{t^2+2}, -\frac{1}{3}e^{-3t}, t - \frac{1}{4}\cos(4t) \right\rangle + \mathbf{C}$$

Outline

Position, speed, velocity, acceleration

Definition 5.

Consider

- A motion $\mathbf{r}(t)$ in $\mathbb{R}^3$ of the form $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$

Then we define

1. Velocity:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

2. Speed:

$$|\mathbf{v}(t)| = \left(x'(t)^2 + y'(t)^2 + z'(t)^2 \right)^{1/2}$$

3. Acceleration:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

Example: circular motion

Motion: We consider

$$\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t) \rangle$$

Velocity:

$$\mathbf{v}(t) = \langle -3\sin(t), 3\cos(t) \rangle$$

Speed:

$$|\mathbf{v}(t)| = 3$$

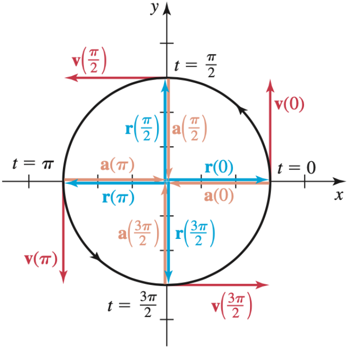
Acceleration:

$$\mathbf{a}(t) = -\langle 3\cos(t), 3\sin(t) \rangle$$

Remarks on circular motion

We have obtained:

1. $\mathbf{r}$ circular motion
2. $\mathbf{v}(t)$ is perpendicular to $\mathbf{r}(t)$
3. Speed is constant
4. $\mathbf{a}(t) = -\mathbf{r}(t)$



Projectile motion (1)

Definition of projectile motion:
Object under the influence of an acceleration $\mathbf{a}(t)$
$\hookrightarrow$ with initial velocity $\mathbf{v}(0)$ and position $\mathbf{r}(0)$

Example: Consider the following situation

- A ball resting on the ground is kicked
  $\hookrightarrow$ with initial velocity $\mathbf{v}(0) = \langle 10, 15, 20 \rangle \mathrm{m/s}$
- Acceleration is only due to gravity

Questions:

1. How long does the ball stay in the air?
2. How far does it fly?
3. How high does it fly?

Projectile motion (2)

Acceleration:

$$\mathbf{a}(t) = \langle 0, 0, -9.8 \rangle \ \text{m/s}^2$$

Velocity:

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,\mathrm{d}t = \langle 0, 0, -9.8t \rangle + \mathbf{C}$$

Velocity with initial condition:

Taking into account $\mathbf{v}(0) = \langle 10, 15, 20 \rangle$ we get

$$\mathbf{v}(t) = \langle 10, 15, -9.8t + 20 \rangle$$

Projectile motion (3)

Motion:

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,\mathrm{d}t = \left\langle 10t, 15t, 20t - 4.9t^2 \right\rangle + \mathbf{D}$$

Motion with initial condition:

Taking into account $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$ we get

$$\mathbf{r}(t) = \left\langle 10t, 15t, 20t - 4.9t^2 \right\rangle$$

Projectile motion (4)

Time of flight:
Until $z(t) = 0$ with $t > 0$. We get

$$t = \frac{20}{4.9} = 4.08\,\mathrm{s}$$

Distance it flies: Given by

$$|\mathbf{r}(4.08)| = \left((40.82)^2 + (61.23)^2\right)^{1/2} \simeq 73.59\,\mathrm{m}$$

Maximal height: Height when $z'(t) = 0$. We have

$$z'(t) = 0 \quad \Longleftrightarrow \quad -9.8t + 20 = 0 \quad \Longleftrightarrow \quad t \simeq 2.04$$

Thus height given by

$$z(2.04) \simeq 20.41$$

Projectile motion (5)

Additional question:
What happens if initial velocity is doubled, ie

$$\mathbf{v}(0) = \langle 20, 30, 40 \rangle$$

Changes on the motion: One can check that

- Time of flight is doubled: $t \simeq 8.16$s
- Distance of flight is quadrupled: $|\mathbf{r}(4.16)| \simeq 294.36$

Outline

Arc length

Definition 6.

We assume

- $\mathbf{r}(t)$ a vector-valued function, $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$
- f', g', h' continuous functions
- Curve $\mathbf{r}$ traversed once on $[a, b]$

Then the arc length of $\mathbf{r}$ between $\mathbf{r}(a)$ and $\mathbf{r}(b)$ is

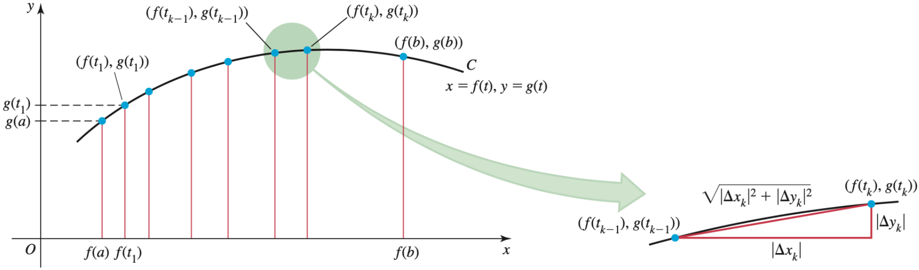
$$L = \int_a^b |\mathbf{r}'(t)| \, \mathrm{d}t.$$

We also have

$$L = \int_a^b \left(f'(t)^2 + g'(t)^2 + h'(t)^2 \right)^{1/2} \mathrm{d}t.$$

Discretized version of arc length

Illustration:



Approximation: We have

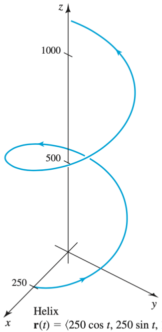
$$L \simeq \sum_k \left(|\Delta x_k|^2 + |\Delta y_k|^2\right)^{1/2} \quad \xrightarrow{k\to\infty} \quad \int_a^b |\mathbf{r}'(t)|\, \mathrm{d}t$$

Flight of an eagle (1)

Situation: An eagle rises at a rate of 100 vertical ft/min on a helical path given by

$$\mathbf{r}(t) = \langle 250 \cos t, 250 \sin t, 100t \rangle$$

Question: How far does the eagle travel in 10 mn?



Helix
$\mathbf{r}(t) = \langle 250 \cos t, 250 \sin t, 100t \rangle$

Flight of an eagle (2)

Speed: We have

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \sqrt{250^2 + 100^2} \simeq 269$$

Length: The distance traveled is

$$L = \int_0^{10} |\mathbf{v}(t)| \, \mathrm{d}t = 2690$$

Arc length function

Theorem 7.

We assume

- $\mathbf{r}(t)$ a vector-valued function, $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$
- f', g', h' continuous functions

Then

1. The arc length function is given by

$$s(t) = \int_a^t |\mathbf{v}(u)| \, \mathrm{d}u.$$

2. If $|\mathbf{v}(u)| = 1$ for all $t \geq a$
$\hookrightarrow$ the parameter t corresponds to arc length.

Helix example (1)

Function: Helix of the form

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 4t \rangle$$

Problem:
Parametrize $\mathbf{r}$ according to its arc length.

Helix example (2)

Velocity:

$$\mathbf{v}(t) = \langle -2\sin(t), 2\cos(t), 4 \rangle$$

Speed: We have

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = 2\sqrt{5}$$

Arc length function: We get

$$s(t) = \int_0^t |\mathbf{v}(u)| \, \mathrm{d}u = 2\sqrt{5}t$$

Helix example (3)

Arc length as parameter: Set $s = 2\sqrt{5}t$.
$\hookrightarrow$ We get a new curve parametrized by s

$$\mathbf{r}_1(s) = \left\langle 2\cos\left(\frac{s}{2\sqrt{5}}\right), 2\sin\left(\frac{s}{2\sqrt{5}}\right), \frac{2s}{\sqrt{5}} \right\rangle$$

Property: For $\mathbf{r}_1$ we have

Increment of Δs in the parameter
$\Longrightarrow$
Increment of Δs in arc length

Outline

Unit tangent vector (reloaded)

Definition 8.

Let

- $\mathbf{r}(t)$ a vector-valued function
- Assume $\mathbf{r}'(t) \neq 0$

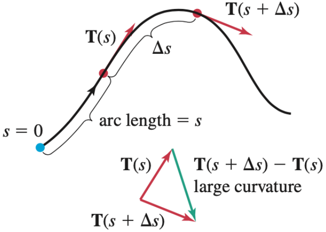
Then the unit tangent vector of $\mathbf{r}$ at time t is defined by

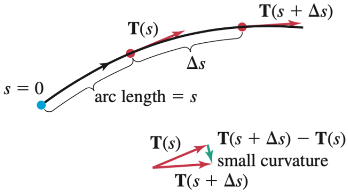
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Intuition of curvature

Idea:

If a curve is curvy, then $\mathbf{T}$ changes quickly with arc length s





Curvature

Definition 9.

Let

- $\mathbf{r}(s)$ a vector-valued function
- Assume $\mathbf{r}$ parametrized by arc length s

Then the curvature of $\mathbf{r}$ at s is defined by

$$\kappa(s) = \frac{\mathrm{d}\mathbf{T}(s)}{\mathrm{d}s}.$$

Problem with the definition:

One cannot always parametrize by s

Curvature formula

Theorem 10.

Let

- $\mathbf{r}(s)$ a vector-valued function
- Assume $\mathbf{r}$ parametrized by t

Then the curvature of $\mathbf{r}$ at time t is given by

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{T}'(t)|}{|\mathbf{v}(t)|}.$$

Curvature: helix example (1)

Function: Helix of the form

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 4t \rangle$$

Problem:
Compute the curvature for $\mathbf{r}$.

Curvature: helix example (2)

Velocity:

$$\mathbf{v}(t) = \langle -2\sin(t), 2\cos(t), 4 \rangle$$

Speed: We have

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = 2\sqrt{5}$$

Unit tangent: We get

$$\mathbf{T}(t) = \frac{1}{2\sqrt{5}} \langle -2\sin(t), 2\cos(t), 4 \rangle$$

Curvature: helix example (3)

Derivative of unit tangent: We have

$$\mathbf{T}'(t) = -\frac{1}{\sqrt{5}} \langle \cos(t), \sin(t), 0 \rangle$$

Curvature: Given by

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{v}(t)|} = \frac{1}{10}.$$

Remarks on curvature

Particular cases:

- Lines have 0 curvature
- Circles have constant curvature

Another formula to compute κ:

$$\kappa(t) = \frac{|\mathbf{r}''(t) \times \mathbf{r}'(t)|}{|\mathbf{r}'(t)|^3}$$