Vector-valued functions

Samy Tindel

Purdue University

Multivariate calculus - MA 261

Mostly taken from *Calculus, Early Transcendentals* by Briggs - Cochran - Gillett - Schulz
Outline

1. Vector-valued functions
2. Calculus of vector-valued functions
3. Motion in space
4. Length of curves
5. Curvature and normal vector
Outline

1. Vector-valued functions
2. Calculus of vector-valued functions
3. Motion in space
4. Length of curves
5. Curvature and normal vector
Functions with values in $\mathbb{R}^3$

**Scalar-valued functions:** We are used to functions like

$$f(t) = 3t^2 + 5 \implies f(1) = 8 \in \mathbb{R}$$

**Vector-valued functions:** In this course we consider

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \implies \mathbf{r}(t) \in \mathbb{R}^3$$
Problem: Consider the line passing through 

\[ P(1, 2, 3) \text{ and } Q(4, 5, 6) \]

Find a vector-valued function for this line
Lines as vector-valued functions (2)

Parallel vector:

\[ \mathbf{v} = (3, 3, 3), \quad \text{simplified as} \quad \mathbf{v} = (1, 1, 1) \]

Equation for the line:

\[ \mathbf{r}(t) = \langle 1 + t, 2 + t, 3 + t \rangle \]

Examples of points:

\[ \mathbf{r}(0) = \langle 1, 2, 3 \rangle, \quad \mathbf{r}(1) = \langle 2, 3, 4 \rangle, \quad \mathbf{r}(2) = \langle 3, 4, 5 \rangle \]
Problem: Graph the curve defined by

\[ r(t) = \left\langle 4 \cos(t), \sin(t), \frac{t}{2\pi} \right\rangle \]
Spiral (2)

Projection on xy-plane: Set $z = 0$. We get

$$\langle 4 \cos(t), \sin(t) \rangle$$

This is an ellipse, counterclockwise, starts at $(4, 0, 0)$

Related surface: We have

$$\frac{x^2}{4} + y^2 = 1$$

Thus curve lies on an elliptic cylinder

Upward direction: The $z$-component is $\frac{t}{2\pi}$

$\implies$ Spiral on the cylinder
Eight loops of the spiral
\[ \mathbf{r}(t) = 4 \cos t \mathbf{i} + \sin t \mathbf{j} + \frac{t}{2\pi} \mathbf{k}, \]
for \(-\infty < t < \infty\)

The spiral lies on the elliptical cylinder
\[ \frac{x^2}{16} + y^2 = 1. \]
Definition: The domain of $t \mapsto \mathbf{r}(t)$ is

$\mapsto$ The intersection of the domains for each component

Example: If

$$\mathbf{r}(t) = \left\langle \sqrt{1 - t^2}, \sqrt{t}, \frac{1}{\sqrt{5 + t}} \right\rangle,$$

then the domain of $\mathbf{r}$ is

$$[0, 1]$$
Outline

1 Vector-valued functions

2 Calculus of vector-valued functions

3 Motion in space

4 Length of curves

5 Curvature and normal vector
Derivative

**Definition 1.**

Let

- \( r(t) \) a vector-valued function
- \( r \) of the form \( r(t) = \langle f(t), g(t), h(t) \rangle \)

Then the derivative of \( r \) is defined by

\[
r'(t) = \lim_{h \to 0} \frac{r(t + h) - r(t)}{h}.
\]

We also have

\[
r'(t) = \langle f'(t), g'(t), h'(t) \rangle
\]
Derivative and velocity

(a) $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$

$\mathbf{r}(t + \Delta t)$

$\mathbf{r}(t)$

$\mathbf{PQ} = \Delta \mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$

(b) As $\Delta t \to 0$, $\frac{\Delta \mathbf{r}}{\Delta t} \to \mathbf{r}'(t)$, which is a tangent vector at $P$. 

Samy T.
Spiral on cone example

Function: Consider the curve defined by

\[ r(t) = \langle t \cos(t), t \sin(t), t \rangle \]

Derivative: We get

\[ r'(t) = \langle -t \sin(t) + \cos(t), t \cos(t) + \sin(t), 1 \rangle \]

Related surface: \( r \) is a spiral on the cone

\[ x^2 + y^2 = z^2 \]
Let $\mathbf{r}(t)$ a vector-valued function

Assume $\mathbf{r}'(t) \neq 0$

Then the unit tangent vector of $\mathbf{r}$ at time $t$ is defined by

$$
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.
$$
Spiral on cone example

**Function:** Consider the curve defined by

\[ r(t) = \langle t \cos(t), t \sin(t), t \rangle \]

**Derivative:** We have seen

\[ r'(t) = \langle -t \sin(t) + \cos(t), t \cos(t) + \sin(t), 1 \rangle \]

**Unit tangent:** We get

\[ T(t) = \left\langle \frac{-t \sin(t) + \cos(t)}{\sqrt{t^2 + 2}}, \frac{t \cos(t) + \sin(t)}{\sqrt{t^2 + 2}}, \frac{1}{\sqrt{t^2 + 2}} \right\rangle \]
Product rules

**Theorem 3.**

Let

- $\mathbf{u}, \mathbf{v}$ vector-valued functions
- $f$ real-valued function

Then we have

$$[f(t)\mathbf{u}(t)]' = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$
$$[\mathbf{u}(t) \cdot \mathbf{v}(t)]' = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$$
$$[\mathbf{u}(t) \times \mathbf{v}(t)]' = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$
Example of product rule

Functions: Consider

\[ r(t) = \langle 1, t, t^2 \rangle, \quad f(t) = e^t \]

Product derivative: We find

\[
\frac{d}{dt} [f(t) r(t)] = e^t \langle 1, t + 1, t^2 + 2t \rangle
\]
Antiderivative

**Definition 4.**

Consider

- $\mathbf{r}$ of the form $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$
- $F, G, H$ antiderivatives of $f, g, h$ respectively
- $\mathbf{R}(t) = \langle F(t), G(t), H(t) \rangle$

Then we have

$$\int \mathbf{r}(t) \, dt = \mathbf{R}(t) + \langle C_1, C_2, C_3 \rangle$$
Example of antiderivative

**Function:** Consider

\[ r(t) = \left\langle \frac{t}{\sqrt{t^2 + 2}}, e^{-3t}, \sin(4t) + 1 \right\rangle \]

**Antiderivative:** We get

\[
\int r(t) \, dt = \left\langle \sqrt{t^2 + 2}, -\frac{1}{3}e^{-3t}, t - \frac{1}{4} \cos(4t) \right\rangle + C
\]
Outline

1. Vector-valued functions
2. Calculus of vector-valued functions
3. Motion in space
4. Length of curves
5. Curvature and normal vector
Consider a motion $r(t)$ in $\mathbb{R}^3$ of the form $r(t) = \langle x(t), y(t), z(t) \rangle$.

Then we define:

1. **Velocity:**
   
   $$v(t) = r'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

2. **Speed:**
   
   $$|v(t)| = \left( x'(t)^2 + y'(t)^2 + z'(t)^2 \right)^{1/2}$$

3. **Acceleration:**
   
   $$a(t) = v'(t) = r''(t)$$
Example: circular motion

**Motion:** We consider

\[ r(t) = \langle 3 \cos(t), 3 \sin(t) \rangle \]

**Velocity:**

\[ v(t) = \langle -3 \sin(t), 3 \cos(t) \rangle \]

**Speed:**

\[ |v(t)| = 3 \]

**Acceleration:**

\[ a(t) = -\langle 3 \cos(t), 3 \sin(t) \rangle \]
Remarks on circular motion

We have obtained:

1. \( \mathbf{r} \) circular motion
2. \( \mathbf{v}(t) \) is perpendicular to \( \mathbf{r}(t) \)
3. Speed is constant
4. \( \mathbf{a}(t) = -\mathbf{r}(t) \)
Definition of projectile motion: 
Object under the influence of an acceleration $\mathbf{a}(t)$ with initial velocity $\mathbf{v}(0)$ and position $\mathbf{r}(0)$

Example: Consider the following situation
- A ball resting on the ground is kicked with initial velocity $\mathbf{v}(0) = \langle 10, 15, 20 \rangle \text{m/s}$
- Acceleration is only due to gravity

Questions:
1. How long does the ball stay in the air?
2. How far does it fly?
3. How high does it fly?
Projectile motion (2)

Acceleration:

\[ a(t) = \langle 0, 0, -9.8 \rangle \text{ m/s}^2 \]

Velocity:

\[ v(t) = \int a(t) \, dt = \langle 0, 0, -9.8t \rangle + C \]

Velocity with initial condition:
Taking into account \[ v(0) = \langle 10, 15, 20 \rangle \] we get

\[ v(t) = \langle 10, 15, -9.8t + 20 \rangle \]
Projectile motion (3)

Motion:

\[ \mathbf{r}(t) = \int \mathbf{v}(t) \, dt = \left\langle 10t, 15t, 20t - 4.9t^2 \right\rangle + D \]

Motion with initial condition:

Taking into account \( \mathbf{r}(0) = \langle 0, 0, 0 \rangle \) we get

\[ \mathbf{r}(t) = \left\langle 10t, 15t, 20t - 4.9t^2 \right\rangle \]
Projectile motion (4)

Time of flight:
Until \( z(t) = 0 \) with \( t > 0 \). We get

\[
t = \frac{20}{4.9} = 4.08 \text{ s}
\]

Distance it flies: Given by

\[
|r(4.08)| = \left( (40.82)^2 + (61.23)^2 \right)^{1/2} \approx 73.59 \text{ m}
\]

Maximal height: Height when \( z'(t) = 0 \). We have

\[
z'(t) = 0 \iff -9.8t + 20 = 0 \iff t \approx 2.04
\]

Thus height given by

\[
z(2.04) \approx 20.41
\]
Additional question: What happens if initial velocity is doubled, ie

\[ \mathbf{v}(0) = \langle 20, 30, 40 \rangle \]

Changes on the motion: One can check that

- Time of flight is doubled: \( t \approx 8.16 \) s
- Distance of flight is quadrupled: \( |\mathbf{r}(4.16)| \approx 294.36 \)
Outline

1. Vector-valued functions
2. Calculus of vector-valued functions
3. Motion in space
4. Length of curves
5. Curvature and normal vector
Arc length

**Definition 6.**

We assume
- \( \mathbf{r}(t) \) a vector-valued function, \( \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \)
- \( f', g', h' \) continuous functions
- Curve \( \mathbf{r} \) traversed once on \([a, b]\)

Then the **arc length** of \( \mathbf{r} \) between \( \mathbf{r}(a) \) and \( \mathbf{r}(b) \) is

\[
L = \int_{a}^{b} |\mathbf{r}'(t)| \, dt.
\]

We also have

\[
L = \int_{a}^{b} \left( f'(t)^2 + g'(t)^2 + h'(t)^2 \right)^{1/2} \, dt.
\]
Discretized version of arc length

Illustration:

Approximation: We have

\[ L \approx \sum_{k} \left( |\Delta x_k|^2 + |\Delta y_k|^2 \right)^{1/2} \quad k \to \infty \quad \int_{a}^{b} |r'(t)| \, dt \]
Flight of an eagle (1)

**Situation:** An eagle rises at a rate of 100 vertical ft/min on a helical path given by

\[ \mathbf{r}(t) = \langle 250 \cos t, 250 \sin t, 100t \rangle \]

**Question:** How far does the eagle travel in 10 mn?
Flight of an eagle (2)

**Speed:** We have

\[ |\mathbf{v}(t)| = |\mathbf{r}'(t)| = \sqrt{250^2 + 100^2} \approx 269 \]

**Length:** The distance traveled is

\[ L = \int_{0}^{10} |\mathbf{v}(t)| \, dt = 2690 \]
We assume
\[ \mathbf{r}(t) \text{ a vector-valued function, } \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \]
\[ f', g', h' \text{ continuous functions} \]

Then
1. The arc length function is given by
   \[ s(t) = \int_{a}^{t} |\mathbf{v}(u)| \, du. \]
2. If \( |\mathbf{v}(u)| = 1 \) for all \( t \geq a \)
   \[ \rightarrow \text{ the parameter } t \text{ corresponds to arc length.} \]
Function: Helix of the form

\[ r(t) = \langle 2 \cos(t), 2 \sin(t), 4t \rangle \]

Problem:
Parametrize \( r \) according to its arc length.
Helix example (2)

Velocity:

\[ \mathbf{v}(t) = \langle -2 \sin(t), 2 \cos(t), 4 \rangle \]

Speed: We have

\[ |\mathbf{v}(t)| = |\mathbf{r}'(t)| = 2\sqrt{5} \]

Arc length function: We get

\[ s(t) = \int_0^t |\mathbf{v}(u)| \, du = 2\sqrt{5}t \]
Helix example (3)

Arc length as parameter: Set \( s = 2\sqrt{5}t \).
\[ \Rightarrow \text{We get a new curve parametrized by } s \]
\[ r_1(s) = \left\langle 2\cos \left( \frac{s}{2\sqrt{5}} \right), 2\sin \left( \frac{s}{2\sqrt{5}} \right), \frac{2s}{\sqrt{5}} \right\rangle \]

Property: For \( r_1 \) we have

Increment of \( \Delta s \) in the parameter
\[ \Rightarrow \]
Increment of \( \Delta s \) in arc length
Outline

1. Vector-valued functions
2. Calculus of vector-valued functions
3. Motion in space
4. Length of curves
5. Curvature and normal vector
Let $\mathbf{r}(t)$ a vector-valued function
Assume $\mathbf{r}'(t) \neq 0$

Then the unit tangent vector of $\mathbf{r}$ at time $t$ is defined by

\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.
\]
Intuition of curvature

Idea:
If a curve is *curvy*, then $\mathbf{T}$ changes *quickly* with arc length $s$.
Curvature

Let \( \mathbf{r}(s) \) a vector-valued function

Assume \( \mathbf{r} \) parametrized by arc length \( s \)

Then the curvature of \( \mathbf{r} \) at \( s \) is defined by

\[
\kappa(s) = \frac{d\mathbf{T}(s)}{ds}.
\]

**Problem with the definition:**
One cannot always parametrize by \( s \)
Curvature formula

**Theorem 10.**

Let

- $\mathbf{r}(s)$ a vector-valued function
- Assume $\mathbf{r}$ parametrized by $t$

Then the curvature of $\mathbf{r}$ at time $t$ is given by

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{T}'(t)|}{|\mathbf{v}(t)|}.$$
Curvature: helix example (1)

Function: Helix of the form

\[ \mathbf{r}(t) = \langle 2 \cos(t), 2 \sin(t), 4t \rangle \]

Problem:
Compute the curvature for \( \mathbf{r} \).
Curvature: helix example (2)

Velocity:

\[ \mathbf{v}(t) = \langle -2\sin(t), 2\cos(t), 4 \rangle \]

Speed: We have

\[ |\mathbf{v}(t)| = |\mathbf{r}'(t)| = 2\sqrt{5} \]

Unit tangent: We get

\[ \mathbf{T}(t) = \frac{1}{2\sqrt{5}} \langle -2\sin(t), 2\cos(t), 4 \rangle \]
Derivative of unit tangent: We have

\[ T'(t) = -\frac{1}{\sqrt{5}} \langle \cos(t), \sin(t), 0 \rangle \]

Curvature: Given by

\[ \kappa(t) = \frac{|T'(t)|}{|v(t)|} = \frac{1}{10}. \]
Remarks on curvature

Particular cases:
- Lines have 0 curvature
- Circles have constant curvature

Another formula to compute $\kappa$:

$$\kappa(t) = \frac{|r''(t) \times r'(t)|}{|r'(t)|^3}$$