MIDTERM 1 - FALL 20

1. A jug of buttermilk is set to cool on a front porch, where the temperature is 0°C. The jug was originally at 87°C. If the buttermilk has cooled to 15°C after 26 minutes, after how many minutes will the jug be at 8°C?

The equation is (see First order slides p. 51)

$$\frac{dT}{dt} = -k(T-\tau) \quad , \quad T(0) = 87 \; , \; k \text{ unknown}$$

Here $\tau = 0$, so the equation becomes

$$\frac{dT}{dt} = -kT \quad \Rightarrow \quad T = 87\, e^{-kt}$$

In order to determine k, let us use the information $T(26) = 15$. We get

$$87\, e^{-26k} = 15 \iff k = \frac{1}{26} \ln\left(\frac{87}{15}\right) \simeq .068$$

Now we wish to find t s.t. $T(t) = 8$. This is expressed as

$$87\, e^{-kt} = 8 \iff t = \underbrace{\boxed{\frac{1}{k}}}_{\simeq 14.79} \ln\left(\frac{87}{8}\right) \simeq 35.29$$

We have to wait for $\boxed{\text{35 minutes}}$

2. A large tank initially contains 20g of salt in 20L of water. A solution containing 6g/L salt flows into the tank at a rate of 4L/min, and the well stirred mixture flows out at the rate of 3L/min. Which of the following differential equations and initial conditions describe the amount of salt A = A(t) in the tank at time t before the tank is full.

We use the equation (see First order, p. 78)

$$\frac{dA}{dt} = \text{rate in} - \text{rate out}$$

Here rate in $= C_{in} \times flow_{in} = 6 \times 4 = 24$

rate out $= C_{out} \times flow_{out}$

$$= \frac{A(t)}{V(t)} \times 3 = \frac{3A}{V(0) + (4-3)t} = \frac{3A}{20+t}$$

Hence the initial value problem is

$$\frac{dA}{dt} = 24 - \frac{3A}{20+t} \quad , \quad A(0) = 20$$

Otherwise stated:

$$\boxed{\frac{dA}{dt} + \frac{3A}{20+t} = 24 \quad , \quad A(0) = 20}$$

Note: This is a linear equation, with integrating factor

$$\mu = e^{3\int \frac{dt}{20+t}} = (20+t)^3$$

Assume that $y = y(x)$ is a solution of the equation

$$\overbrace{(3x^2 + y)}^{M}\, dx + \overbrace{(x + 2y)}^{N}\, dy = 0$$

and $y(1) = 2$. What is the value of $y(2)$?

We have

$M_y = 1$, $M_x = 1$

Hence the equation is <u>exact</u>

In order to compute the corresponding ϕ, we follow the recipe on First order p. 105. Hence

$$\phi(x,y) = \int M\, dx = \int (3x^2 + y)\, dx$$
$$= x^3 + yx + h(y)$$

Then we identify h by writing

$$\phi_y = N \iff x + h'(y) = x + 2y$$
$$\iff h'(y) = 2y \iff h(y) = y^2 + c,$$

Thus the general solution to the equation is

$$x^3 + yx + y^2 = c$$

The initial condition $x=1, y=2$ yields $c = 7$, so that the implicit form of the solution is

$$\boxed{x^3 + yx + y^2 = 7}$$

One can solve for y by writing

$$\left(y+\frac{x}{2}\right)^2 = 7 - x^3 + \frac{x^2}{4}$$

$$\Leftrightarrow \quad y = \frac{-x}{2} \pm \left(7 - x^3 + \frac{x^2}{4}\right)^{\frac{1}{2}}$$

The sign is determined by plugging the initial condition $y(1) = 2$ again. We thus choose the $+$ sign. We obtain

$$y = \frac{-x}{2} + \left(7 - x^3 + \frac{x^2}{4}\right)^{\frac{1}{2}}$$

For $x = 2$, this yields

$$\boxed{y(2) = -1}$$

4. If the function f(x,y) is continuous near the point (a,b), then at least one solution of the differential equation $y' = f(x,y)$ exists on some open interval I containing the point $x = a$ and, moreover, that if in addition the partial derivative $\frac{\partial f}{\partial y}$ is continuous near (a,b) then this solution is unique on some (perhaps smaller) interval J. Determine whether existence of at least one solution of the given initial value problem is thereby guaranteed and, if so, whether uniqueness of that solution is guaranteed.

$$\frac{dy}{dx} = \sqrt{x-y};\ y(1) = 1$$

Here $f(x,y) = \sqrt{x-y}$, and we need to establish the continuity of f near the point

$$(x,y) = (1,1)$$

As a function in $\mathbb{R}^2$, f is continuous (as well as f_y) wherever $x-y > 0$. This is not the case for $(x,y) = (1,1)$, since in this case $x-y = 0$. Hence existence and uniqueness are not guaranteed by the theorem

Let y = y(x) satisfy the following initial value problem

$$\frac{dy}{dx} + \frac{2}{x} y = 8x^2 \sqrt{y}$$

y(1) = 4.

What is the value of $y(\sqrt{2})$?

This is a Bernoulli equation. The standard method is explained in First order p. 96 :

① Write the equation as

$$2\times \frac{1}{2} y^{-1/2} y' + \frac{2}{x} y^{1/2} = 8x^2$$

② Set $y^{1/2} = u$, so that $u' = \frac{1}{2} y^{-1/2} y'$. We get

$$2u' + \frac{2}{x} u = 8x^2 \iff u' + \frac{1}{x} u = 4x^2$$

③ Solve the linear equation : the integrating factor is $\mu = e^{\int dx/x} = x$. We get

$$(xu)' = 4x^3 \iff xu = x^4 + c$$

$$\iff u = x^3 + \frac{c}{x}$$

The initial value is $u(1) = \sqrt{y(1)} = \sqrt{4} = 2$, so that $c = 1$. Thus

$$u = x^3 + \frac{1}{x} \quad , \quad y = u^2 = \left(x^3 + \frac{1}{x}\right)^2$$

For $x = \sqrt{2}$ we get $\left(2\sqrt{2} + \frac{\sqrt{2}}{2}\right)^2 = 2 \times \left(\frac{5}{2}\right)^2$

Hence $\boxed{y(\sqrt{2}) = \frac{25}{2}}$

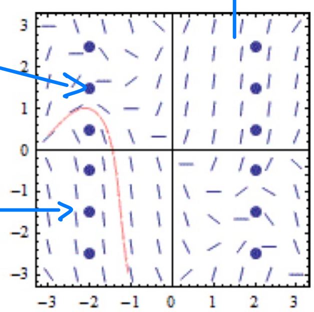
No specific direction for the slope

Here $\sin x > 0$, $\sin y > 0$, thus slope of the form $\nearrow$

6. The slope field of the indicated differential equation has been provided, together with a solution curve. Sketch solution curves through the additional points marked in the slope field.

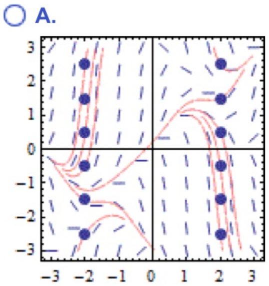
$$\frac{dy}{dx} = 5\,\mathbf{sin}\,x + 5\,\mathbf{sin}\,y$$

Here $\sin x < 0$, $\sin y < 0$, thus slope of the form $\searrow$

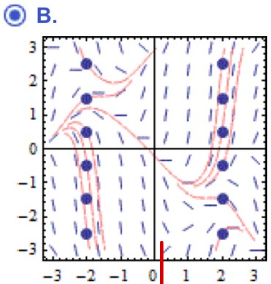


Choose the correct graph below.

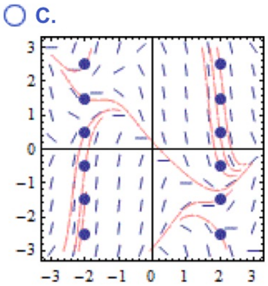
A.



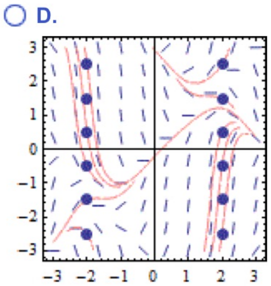
B.



C.



D.



This is the only slope field with the same behavior as above

7. A Las Vegas casino tells their customers who want to play poker that C(t), the amount of cash a poker player has at time t after they start playing, satisfies the differential equaton

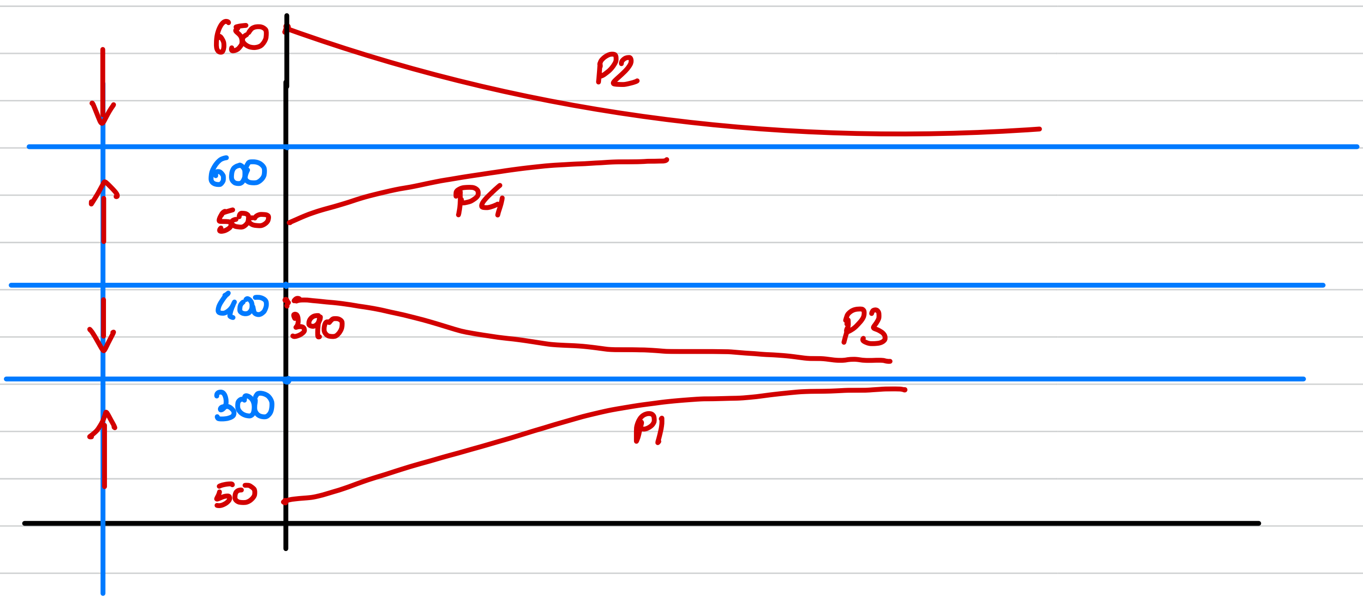
$$\frac{dC(t)}{dt} = (C(t) - 300)(C(t) - 400)(600 - C(t))$$

There are four players playing the game, P1, P2, P3 and P4. If C(0) is the amount of money the gambler brings to the table, P1 brings \$50, P2 brings \$650, P3 brings \$390 and P4 brings \$500, which of the following is correct if the players keep playing at the same poker game for a very long time?

We will use a phase diagram with

$f(c) = (c-300)(c-400)(600-c)$

The critical values are $c = 300, 400, 600$.



Hence in the long run:

P1 wins \$ 250

P2 loses \$ 50 $\Rightarrow$ $\boxed{\text{P1 wins the most money; P3 loses the most money}}$

P3 loses \$ 90

P4 wins \$ 100

If $y = y(t)$ is the solution of the initial value problem

$t y' - y = t^2 e^{-t}$,

$y(1)=3$,

what is the value of $y(3)$?

This is a linear equation. Its standard form is

$$y' - \frac{1}{t} y = t e^{-t}$$

The integrating factor is $\mu = e^{-\int \frac{1}{t} dt} = t^{-1}$.

We obtain

$$\left(\frac{1}{t} y\right)' = e^{-t}$$

Integrate on both sides. We get

$$\frac{1}{t} y = -e^{-t} + c \iff y = -t e^{-t} + ct$$

The initial value is $y(1) = 3$. This yields

$$c = 3 + e^{-1} \implies y = (3 + e^{-1}) t - t e^{-t}$$

Hence

$$\boxed{y(3) = 9 + 3e^{-1} - 3e^{-3}}$$

Find the explicit particular solution of the differential equation for the initial value provided.

$$\frac{dy}{dx} = 5x^2 y - y, \; y(1) = -3$$

The equation can be seen as either linear or separable. We choose separable. We thus write it as

$$\frac{dy}{y} = (5x^2 - 1)\, dx$$

Integrate on both sides to get

$$\ln(y) = \frac{5}{3} x^3 - x + c_1$$

$$\Leftrightarrow \quad y = c_2 \exp\left(\frac{5}{3} x^3 - x\right)$$

The initial data is $y(1) = -3$, which yields

$$c_2 \exp\left(\frac{5}{3} - 1\right) = -3 \Leftrightarrow c_2 = -3 \exp\left(-\frac{2}{3}\right)$$

We end up with

$$y = -3 \exp\left(\frac{5}{3} x^3 - x - \frac{2}{3}\right)$$

10. A population of tilapias in a pond, denoted by x=x(t), where t is counted in years, obeys the following differential equation

$$\frac{dx}{dt} = 1200\,x - x^2$$

If the initial population was x(0) = 2500 tilapias, what will be the time T until half of the tilapias die? What will be the population of tilapias in the pond after 10 years?

Write $\dfrac{dx}{(x-1200)x} = -dt$ (1)

Then $\dfrac{1}{(x-1200)x} = \dfrac{1}{1200}\left(\dfrac{1}{x-1200} - \dfrac{1}{x}\right)$

We integrate on both sides of (1) and get

$$\ln\left(\frac{x-1200}{x}\right) = -1200\,t + c_1$$

When $t=0$ we have $x = 2500$, so that

$c_1 = \ln\left(\frac{13}{25}\right) \Rightarrow \ln\left(\frac{x-1200}{x}\right) = \ln\left(\frac{13}{25}\right) - 1200\,t$ (2)

(i) We want to know T such that $x = 1250$. Plugging into (2) we get

$$-\ln(25) = \ln(13/25) \quad 1200\,T$$

$$\Leftrightarrow \boxed{T = \frac{1}{1200}\ln(13)}$$

(i) Write (2) as $\dfrac{x-1200}{x} = \dfrac{13}{25}e^{-1200t}$

$$\Leftrightarrow x = \frac{1200}{1 - \frac{13}{25}e^{-1200t}}$$

For $t = 10$, we get $\boxed{x \simeq 1200}$

11. Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x}{2(x+y)},$$

which is satisfied for $x > 0$. Suppose a solution $y(x)$ satisfies $y(1) = 1$. What is the value of $y(e^5)$?

This is a homogeneous equation. As a function of $v = y/x$, the rhs of the equation reads

$$F(v) = v + \frac{1}{2(1+v)}$$

According to Thm 7 p. 87 in First order, the separable equation for v is

$$\frac{1}{F(v)-v}\,dv = \frac{dx}{x} \iff 2(1+v)\,dv = \frac{dx}{x}$$

Integrate on both sides. This yields

$$2v + v^2 = \ln(x) + c_1 \quad \text{here } x>0$$

$$\iff (v+1)^2 = \ln(x) + \underbrace{c_1 + 1}_{=c_2}$$

$$\iff v = -1 \pm (\ln(x) + c_2)^{\frac{1}{2}}$$

$y = xv$

$$\iff y = -x \pm x\,(\ln(x) + c_2)^{\frac{1}{2}}$$

The initial condition is $y(1) = 1$. This forces the + sign and we get $1 = -1 + c_2^{\frac{1}{2}} \iff c_2 = 4$

Hence $y(x) = x\left((\ln(x)+4)^{\frac{1}{2}} - 1\right)$

Thus $\boxed{y(e^5) = 2e^5}$