Eigenvalues and eigenvectors

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Differential equations and linear algebra - MA 262

Taken from *Differential equations and linear algebra*
Edwards, Penney, Calvis

Outline

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David Hilbert

Hilbert:

- Lived 1862-1943 in Germany
- Among top 3 mathematicians of 20th century
- Foundations of mathematics
- Infinite dimensional vector spaces
  $\hookrightarrow$ Hilbert spaces
- Number theory
- Axioms of geometry
- Coined the term eigenvalue

Eigenvalue: In German, 'own' value (or proper value).

Definitions

Definition 1.

Let

- A be a $n \times n$ matrix
- $\lambda \in \mathbb{R}$

Then

1. If the following system has nontrivial solutions:

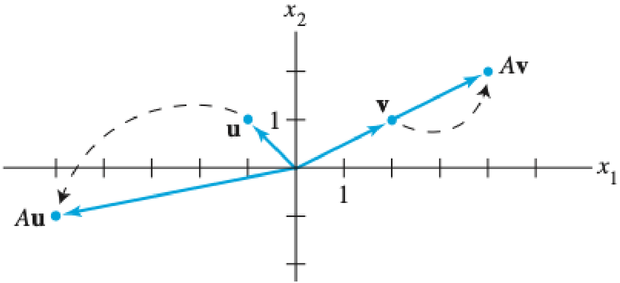
$$A\mathbf{v} = \lambda\,\mathbf{v}, \tag{1}$$

we say that λ is an eigenvalue of A.

2. If λ is an eigenvalue
$\hookrightarrow$ A vector $\mathbf{v}$ satisfying (1) is called eigenvector.

Illustration

Illustration in $\mathbb{R}^2$:



Example of eigenvector

Matrix:

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \tag{2}$$

Aim: Show that 7 is an eigenvalue
$\hookrightarrow$ Can be reduced to show that $A - 7I = \mathbf{0}$ has nontrivial solution

Row echelon form:

$$(A - 7I)^\sharp = \begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

General form of eigenvectors for $\lambda = 7$:

$$t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{with} \quad t \in \mathbb{R}$$

Independence of eigenvectors

Theorem 2.

Let

- A a $n \times n$ matrix
- $\lambda_1, \ldots, \lambda_r$ distinct eigenvalues
- $\mathbf{v}_1, \ldots, \mathbf{v}_r$ corresponding eigenvectors

Then

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent

Consequence:

A cannot have more than n distinct eigenvalues

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Strategy for the eigenvalue problem

Strategy:

1. The eigenvalues are scalars such that

$$\det(A - \lambda I) = 0$$

2. If the eigenvalues $\lambda_1, \ldots, \lambda_k$ are given by Step 1,
$\hookrightarrow$ The eigenvectors solve the systems $(i = 1, \ldots, k)$

$$(A - \lambda_i)\, \mathbf{v}_i = \mathbf{0},$$

Example with simple eigenvalues

Matrix:

$$A = \begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix} \tag{3}$$

Characteristic equation:

$$\begin{vmatrix} 5-\lambda & -4 \\ 8 & -7-\lambda \end{vmatrix} = 0 \quad \Longleftrightarrow \quad \lambda^2 + 2\lambda - 3 = 0$$

Eigenvalues:

$$\lambda_1 = -3, \qquad \lambda_2 = 1$$

Example with simple eigenvalues (2)

Computation of $A - \lambda_1 I$: We get

$$A + 3I = \begin{bmatrix} 8 & -4 \\ 8 & -4 \end{bmatrix}$$

Reduced row-echelon form:

$$A + 3I \sim \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

Eigenvectors for $\lambda_1 = -3$:

$$\{r\,\mathbf{v}_1;\ r \in \mathbb{R}\}, \quad \text{where} \quad \mathbf{v}_1 = (1, 2)$$

Example with simple eigenvalues (3)

Eigenvectors for $\lambda_2 = 1$:

$$\{r\,\mathbf{v}_2;\ r \in \mathbb{R}\}, \quad \text{where} \quad \mathbf{v}_2 = (1, 1)$$

Remark:

The family $\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a basis of $\mathbb{R}^2$.

Remark 2: If $\mathbf{x} = c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2$, then

$$A^k\mathbf{x} = c_1\,\lambda_1^k\,\mathbf{v}_1 + c_2\,\lambda_2^k\,\mathbf{v}_2$$

Example with double eigenvalue

Matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{4}$$

Characteristic equation:

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0 \quad \Longleftrightarrow \quad (\lambda - 1)^2 = 0$$

Eigenvalues:

$$\lambda_1 = 1, \qquad \text{repeated twice}$$

Example with double eigenvalues (2)

Computation of $A - \lambda_1 I$: We get directly a reduced row-echelon form

$$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Eigenvectors for $\lambda_1 = 1$: We get only one linearly indep. eigenvector

$$\{r\,\mathbf{v}_1;\ r \in \mathbb{R}\}, \quad \text{where} \quad \mathbf{v}_1 = (1, 0)$$

Conclusion:

- Number of indep. eigenvectors is less than order of eigenvalue
- The matrix A is said to be defective

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Complex eigenvalues

Theorem 3.

Let A be such that

- A is an $n \times n$ matrix
- A has real-valued elements
- A admits a complex eigenvalue λ with eigenvector $\mathbf{v}$

Then

$$\bar{\lambda} \text{ is an eigenvalue for } A \text{ with eigenvector } \bar{\mathbf{v}}.$$

Example with complex eigenvalues

Matrix:

$$A = \begin{bmatrix} -2 & -6 \\ 3 & 4 \end{bmatrix}$$

Characteristic polynomial:

$$p(\lambda) = \begin{vmatrix} -2-\lambda & -6 \\ 3 & 4-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 10$$

Eigenvalues:

$$\lambda_1 = 1 + 3\imath, \qquad \lambda_2 = \bar{\lambda}_1 = 1 - 3\imath$$

Example with complex eigenvalues (2)

Computation of $A - \lambda_1 I$: We get

$$A - (1+3\imath)I = \begin{bmatrix} -3-3\imath & -6 \\ 3 & 3-3\imath \end{bmatrix}$$

Reduced row-echelon form:

$$(A - (1+3\imath)I)^{\sharp} \sim \begin{bmatrix} 1 & 1-\imath & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors for $\lambda_1 = 1 + 3\imath$:

$$\{r\,\mathbf{v}_1;\ r \in \mathbb{R}\}, \quad \text{where} \quad \mathbf{v}_1 = (-(1-\imath), 1)$$

Example with complex eigenvalues (3)

Eigenvectors for $\lambda_2 = 1 - 3\imath$:

$$\{r\,\mathbf{v}_2;\ r \in \mathbb{R}\}, \quad \text{where} \quad \mathbf{v}_2 = \bar{\mathbf{v}}_1 = (-(1+\imath), 1)$$

Remark:

The family $\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a basis of $\mathbb{C}^2$.