Linear equations in linear algebra

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Differential equations and linear algebra - MA 262

Taken from *Differential equations and linear algebra*
Pearson Collections
Outline

1. Systems of linear equations
2. Row reduction and echelon form
3. Vector equations
4. Matrix equation $Ax = b$
5. Solution sets of linear systems
6. Linear dependence
7. Introduction to linear transformations
8. The matrix of a linear transformation
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Systems of linear equations

General form of a $m \times n$ linear system:

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
  \vdots \\
  a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m 
\end{align*}
\]  

(1)

System coefficients: The real numbers $a_{ij}$

System constants: The real numbers $b_i$

Homogeneous system: When $b_i = 0$ for all $i$
Example of linear system

Linear system in $\mathbb{R}^3$:

$$\begin{align*}
  x_1 + x_2 + x_3 &= 1 \\
  x_2 - x_3 &= 2 \\
  x_2 + x_3 &= 6
\end{align*}$$

Unique solution by substitution:

$$x_1 = -5, \quad x_2 = 4, \quad x_3 = 2$$
Notation $\mathbb{R}^n$

**Definition of $\mathbb{R}^n$:**
Set of ordered $n$-uples of real numbers $(x_1, \ldots, x_n)$

**Matrix notation:**
An element of $\mathbb{R}^n$ can be seen as a row or column $n$-vector

$$(x_1, \ldots, x_n) \leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \leftrightarrow [x_1, \ldots, x_n]$$
Related matrices

**Matrix of coefficients:** For the system (1), given by

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
    a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}
\]

**Augmented matrix:** For the system (1), given by

\[
A^\# = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\
    a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m
\end{bmatrix}
\]
Vector formulation

Example of system in $\mathbb{R}^3$:

\[
\begin{align*}
  x_1 + 3x_2 - 4x_3 &= 1 \\
  2x_1 + 5x_2 - x_3 &= 5 \\
  x_1 + 6x_3 &= 3
\end{align*}
\]

Related matrices:

\[
A = \begin{bmatrix}
  1 & 3 & -4 \\
  2 & 5 & -1 \\
  1 & 0 & 6
\end{bmatrix}, \quad b = \begin{bmatrix}
  1 \\
  5 \\
  3
\end{bmatrix}, \quad x = \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
\]

Augmented matrix:

\[
A^\# = \begin{bmatrix}
  1 & 3 & -4 & 1 \\
  2 & 5 & -1 & 5 \\
  1 & 0 & 6 & 3
\end{bmatrix}
\]
Linear systems in $\mathbb{R}^3$

General system in $\mathbb{R}^3$:

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
    a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{align*}
\]

Geometrical interpretation: Intersection of 3 planes in $\mathbb{R}^3$
Sets of solutions

Possible sets of solutions: In the general $\mathbb{R}^n$ case we can have

- No solution to a linear system
- A unique solution
- An infinite number of solutions

**Definition 1.**

Consider a linear system given by (1). Then

1. If there is at least one solution the system is **consistent**
2. If there is no solution the system is **inconsistent**
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Elementary row operations

Operations leaving the system unchanged:

1. Permute equations
2. Multiply a row by a nonzero constant
3. Add a multiple of one equation to another equation

Example of system:

\[
\begin{align*}
    x_1 & + 3x_2 & - 4x_3 & = 1 \\
2x_1 & + 5x_2 & - x_3 & = 5 \\
    x_1 & & + 6x_3 & = 3
\end{align*}
\]
Example of elementary row operations

Example of system:

\[
\begin{align*}
    x_1 + 3x_2 - 4x_3 &= 1 \\
    2x_1 + 5x_2 - x_3 &= 5 \\
    x_1 + 6x_3 &= 3
\end{align*}
\]

Permute \( R_1 \) and \( R_2 \): Denoted by \( P_{12} \)

\[
\begin{align*}
    2x_1 + 5x_2 - x_3 &= 5 \\
    x_1 + 3x_2 - 4x_3 &= 1 \\
    x_1 + 6x_3 &= 3
\end{align*}
\]
Example of elementary row operations (2)

Example of system:

\[
\begin{align*}
    x_1 + 3x_2 - 4x_3 &= 1 \\
    2x_1 + 5x_2 - x_3 &= 5 \\
    x_1 + 6x_3 &= 3
\end{align*}
\]

Multiply \( R_2 \) by \(-2\): Denoted by \( M_2(-2) \)

\[
\begin{align*}
    2x_1 + 5x_2 - x_3 &= 5 \\
    -4x_1 - 10x_2 + 2x_3 &= -10 \\
    x_1 + 6x_3 &= 3
\end{align*}
\]
Example of elementary row operations (3)

Example of system:

\[
\begin{align*}
   x_1 & + 3x_2 & - 4x_3 & = 1 \\
   2x_1 & + 5x_2 & - x_3 & = 5 \\
   x_1 & & + 6x_3 & = 3
\end{align*}
\]

Add 2R_3 to R_1: Denoted by A_{31}(2)

\[
\begin{align*}
   3x_1 & + 3x_2 & + 8x_3 & = 7 \\
   2x_1 & + 5x_2 & - x_3 & = 5 \\
   x_1 & & + 6x_3 & = 3
\end{align*}
\]
Elementary operations in matrix form

Example of system:

\[
\begin{align*}
    x_1 + 3x_2 - 4x_3 &= 1 \\
    2x_1 + 5x_2 - x_3 &= 5 \\
    x_1 + 6x_3 &= 3
\end{align*}
\]

Adding \(2R_3\) to \(R_1\):

\[
\begin{pmatrix}
    1 & 3 & -4 & 1 \\
    2 & 5 & -1 & 5 \\
    1 & 0 & 6 & 3
\end{pmatrix}
\begin{pmatrix}
    & & & A_{31}(2)
\end{pmatrix}
\begin{pmatrix}
    3 & 3 & 8 & 7 \\
    2 & 5 & -1 & 5 \\
    1 & 0 & 6 & 3
\end{pmatrix}
\]
A $m \times n$ matrix is row-echelon whenever

1. If there are rows consisting only of 0’s
   $\rightarrow$ They are at the bottom of the matrix

2. 1st nonzero entries of each row have a triangular shape
   $\rightarrow$ Called leading entries

3. All entries in a column below a leading entry are 0’s
A reduced row-echelon matrix is a matrix $A$ such that

1. $A$ is row-echelon
2. All leading entries are $= 1$
3. Any column with a leading 1 has zeros everywhere else
Example of row-echelon matrix

Row-echelon matrix:

\[
\begin{bmatrix}
1 & 1 & -1 & 4 \\
0 & 1 & -3 & 5 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

Related system:

\[
\begin{align*}
x_1 + x_2 - x_3 &= 4 \\
x_1 + x_2 - 3x_3 &= 5 \\
x_3 &= 2
\end{align*}
\]

Solution of the system: very easy thanks to a back substitution

\[
x_3 = 2, \quad x_2 = 11, \quad x_1 = -5
\]

Strategy to solve general systems:

\[\leftarrow\text{Reduction to a row-echelon system}\]
Reduction to a row-echelon system

Algorithm:

1. Start with a $m \times n$ matrix $A$. If $A = 0$, go to step 7.
2. Pivot column: leftmost nonzero column.
   Pivot position: topmost position in the pivot column.
3. Use elementary row operations to put 1 in the pivot position.
4. Use elementary row operations to put zeros below pivot position.
5. If all rows below pivot are 0, go to step 7.
6. Otherwise apply steps 2 to 5 to the rows below pivot position.
7. The matrix is row-echelon.
Example of reduction

First operations:

\[
\begin{bmatrix}
3 & 2 & -5 & 2 \\
1 & 1 & -1 & 1 \\
1 & 0 & -3 & 4
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 & 1 \\
3 & 2 & -5 & 2 \\
1 & 0 & -3 & 4
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & -1 & -2 & -1 \\
0 & -1 & -2 & 3
\end{bmatrix}
\]

\[A_{12}(-3), A_{13}(-1)\]

Iteration:

\[M_2(-1)\]

\[
\begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & 1 & 2 & 1 \\
0 & -1 & -2 & 3
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & 1 & 2 & 1 \\
0 & 0 & 0 & 4
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & 1 & 2 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

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Linear equations
Differential equations
Reduced row-echelon matrices

**Definition 4.**

A reduced row-echelon matrix is a matrix $A$ such that

1. $A$ is row-echelon
2. All leading entries are $= 1$
3. Any column with a leading 1 has zeros everywhere else
Example of reduced row-echelon

Example:

\[
\begin{bmatrix}
1 & 9 & 26 \\
0 & 14 & 28 \\
0 & -14 & -28
\end{bmatrix}
\overset{M_2(1/14)}{\sim}
\begin{bmatrix}
1 & 9 & 26 \\
0 & 1 & 2 \\
0 & -14 & -28
\end{bmatrix}
\overset{A_{21}(-9), A_{23}(14)}{\sim}
\begin{bmatrix}
1 & 0 & 8 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
\]

Remark: One can also

1. Obtain the row-echelon form
2. Get the reduced row-echelon working upward and to the left
Basic idea to solve linear systems

**Aim:**
Solve a linear system of equations

**Strategy:**
1. Reduce the augmented matrix to row-echelon
2. Solve the system backward thanks to the row-echelon form
Example of system (1)

System:

\[
\begin{align*}
3x_1 - 2x_2 + 2x_3 &= 9 \\
x_1 - 2x_2 + x_3 &= 5 \\
2x_1 - x_2 - 2x_3 &= -1
\end{align*}
\]

Augmented matrix:

\[
A^\# = \begin{bmatrix}
3 & -2 & 2 & 9 \\
1 & -2 & 1 & 5 \\
2 & -1 & -2 & -1
\end{bmatrix}
\]
Example of system (2)

Row-echelon form of the augmented matrix:

\[
A^\# \sim \begin{bmatrix}
1 & -2 & 1 & 5 \\
0 & 1 & 3 & 5 \\
0 & 0 & 1 & 2 \\
\end{bmatrix}
\]

System corresponding to the augmented matrix:

\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 5 \\
x_2 + 3x_3 &= 5 \\
x_3 &= 2
\end{align*}
\]

Solution set:

\[
S = \{(1, -1, 2)\}
\]
Solving with reduced row-echelon

System:

\[
\begin{align*}
x_1 + 2x_2 - x_3 &= 1 \\
2x_1 + 5x_2 - x_3 &= 3 \\
x_1 + 3x_2 + 2x_3 &= 6
\end{align*}
\]

Augmented matrix:

\[
A^\# = \begin{bmatrix}
1 & 2 & -1 & 1 \\
2 & 5 & -1 & 3 \\
1 & 3 & 2 & 6
\end{bmatrix}
\]
Solving with reduced row-echelon (2)

Reduced row-echelon form of the augmented matrix:

\[
A^\# \sim \begin{bmatrix}
1 & 0 & 0 & 5 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 2 \\
\end{bmatrix}
\]

System corresponding to the augmented matrix:

\[
\begin{align*}
x_1 & = 5 \\
x_2 & = -1 \\
x_3 & = 2 \\
\end{align*}
\]

Solution set:

\[
S = \{(5, -1, 2)\}
\]
Comparison between reduced and non reduced

Pros and cons:
- For Gauss-Jordan, the final backward system is easier to solve
- The reduced row-echelon form is costly in terms of computations
- Overall, Gauss is more efficient than Gauss-Jordan for systems

Main interest of Gauss-Jordan:
- One can compute the inverse of a matrix
Ex. of system with $\infty$ number of solutions

System:

\[
\begin{align*}
x_1 & -2x_2 + 2x_3 - x_4 = 3 \\
3x_1 & + x_2 + 6x_3 + 11x_4 = 16 \\
2x_1 & - x_2 + 4x_3 + 4x_4 = 9
\end{align*}
\] (2)

Row-echelon form of the augmented matrix:

\[
A^\# \sim \begin{bmatrix}
1 & -2 & 2 & -1 & 3 \\
0 & 1 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Ex. of system with $\infty$ number of solutions (2)

**Consistency:** From row-echelon form we have

- 4 variables
- 2 leading entries
- No equation of the form $1 = 0$

Therefore we have a

$\rightarrow$ **Consistent system with infinite number of solutions**
Ex. of system with $\infty$ number of solutions (3)

Rule for systems with $\infty$ number of solutions:
- Choose as free variables those variables that **do not** correspond to a leading 1 in row-echelon form of $A^\#$

Application to system (2):
- Free variables: $x_3 = s$ and $x_4 = t$

Solution set:

$$S = \{(5 - 2s - 3t, 1 - 2t, s, t); \ s, t \in \mathbb{R}\}$$

$$= \{(5, 1, 0, 0) + s (-2, 0, 1, 0) + t (-3, -2, 0, 1); \ s, t \in \mathbb{R}\}$$
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Vectors

Fact: A tuple

\[ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \]

can be geometrically interpreted as a vector

Illustration of an addition:
Algebraic operations on vectors (1)

1. **Commutativity of addition:** For all $u, v \in \mathbb{R}^n$,

   $$u + v = v + u$$

2. **Associativity of addition:** For all $u, v, w \in \mathbb{R}^n$,

   $$(u + v) + w = u + (v + w)$$

3. **Existence of a zero vector:** There exists $0 \in \mathbb{R}^n$ such that

   $$v + 0 = v$$

4. **Existence of additive inverses in $\mathbb{R}^n$:**
   For all $v \in \mathbb{R}^n$, there exists $-v \in \mathbb{R}^n$ such that

   $$v + (-v) = 0$$
Algebraic operations on vectors (2)

5. **Unit property:** For all \( v \in \mathbb{R}^n \), we have

\[
1v = v
\]

6. **Associativity of scalar multiplication:**

\[
(r s)v = r(s v)
\]

7. **Distributivity 1:**

\[
r(u + v) = ru + rv
\]

8. **Distributivity 2:**

\[
(r + s)v = rv + sv
\]
Linear combinations

Definition 5.

Let

- \( \mathbf{v}_1, \ldots, \mathbf{v}_p \) family of vectors in \( \mathbb{R}^n \).
- \( c_1, \ldots, c_p \) family of scalars in \( \mathbb{R} \)

Then a linear combination of \( \mathbf{v}_1, \ldots, \mathbf{v}_p \) with weights \( c_1, \ldots, c_p \) is given by

\[
y = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p.
\]
Example in $\mathbb{R}^2$

Example: Consider the vectors of $\mathbb{R}^2$:

\[ \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]

Then

\[ 2\mathbf{v}_1 - \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \]

Set of linear combinations:
Spanning subset of $\mathbb{R}^n$

**Definition 6.**

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ family of vectors in $\mathbb{R}^n$.

Consider the collection of $\mathbf{x} \in \mathbb{R}^n$ which can be written as a linear combination

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k,$$

we say that

$$\mathbf{x} \in \text{Span} \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$$
Example in $\mathbb{R}^3$

Illustration of spans:

$\text{Span}\{v\}$

$\text{Span}\{u, v\}$
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Matrix

**Definition 7.**

A $m \times n$ matrix is a rectangular array of numbers with $m$ rows and $n$ columns.

Example of $2 \times 3$ matrix:

$$A = \begin{bmatrix} \frac{3}{2} & \frac{2}{3} & \frac{1}{5} \\ 0 & \frac{5}{4} & -\frac{3}{7} \end{bmatrix}$$
Index notation

Recall: we have

\[ A = \begin{bmatrix} \frac{3}{2} & \frac{2}{3} & \frac{1}{5} \\ 2 & \frac{3}{4} & -\frac{3}{7} \end{bmatrix} \]

Index notation: For the matrix \( A \) we have

\[ a_{12} = \frac{2}{3}, \quad a_{21} = 0, \quad a_{23} = -\frac{3}{7} \]

Index notation for a \( m \times n \) matrix:

\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix} \]
Vectors

A row 3-vector:

\[ a = \begin{bmatrix} \frac{2}{3} & -\frac{1}{5} & \frac{4}{7} \end{bmatrix} \]

A column 5-vector:

\[ b = \begin{bmatrix} 1 \\ 4 \\ \pi \\ -67 \\ 3 \end{bmatrix} \]
Multiplication as a linear combination

**Definition 8.**

Let

- $A$ a $m \times n$ matrix
- $x \in \mathbb{R}^n$

Then

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1a_1 + \cdots + x_na_n$$
Example

Matrices:

\[
A = \begin{bmatrix}
1 & -2 \\
0 & 2
\end{bmatrix}
\quad \text{and} \quad
x = \begin{bmatrix}
2 \\
1
\end{bmatrix}
\]

Product:

\[
Ax = \begin{bmatrix}
0 \\
2
\end{bmatrix}
\]
Rules for multiplications

Rules to follow:

\[ A(u + v) = Au + Av \quad \text{Distributive law} \]
\[ A(cu) = c(Au) \quad \text{Associative law} \]
\[ A0 = 0 \quad \text{Absorbing state} \]
Vector formulation

Example of system in $\mathbb{R}^3$:

\[
\begin{align*}
    x_1 + 3x_2 - 4x_3 &= 1 \\
    2x_1 + 5x_2 - x_3 &= 5 \\
    x_1 + 6x_3 &= 3
\end{align*}
\]

Related matrices:

\[
A = \begin{bmatrix}
    1 & 3 & -4 \\
    2 & 5 & -1 \\
    1 & 0 & 6
\end{bmatrix}, \quad b = \begin{bmatrix}
    1 \\
    5 \\
    3
\end{bmatrix}, \quad x = \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix}
\]

Vector formulation of the system:

\[Ax = b\]
Existence of solutions

Theorem 9.

Let

- $A$ a $m \times n$ matrix

Then the following statements are equivalent

1. For each $\mathbf{b} \in \mathbb{R}^m$, the system $A\mathbf{x} = \mathbf{b}$ has a solution
2. Each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of columns of $A$
3. The columns of $A$ span $\mathbb{R}^m$
4. $A$ has a pivot in every row
Example

Matrix of a generalized system:

\[ A\# = \begin{bmatrix}
1 & 3 & 4 & b_1 \\
-4 & 2 & -6 & b_2 \\
-3 & -2 & -7 & b_3 \\
\end{bmatrix} \]

Row-echelon form:

\[ A\# \sim \begin{bmatrix}
1 & 3 & 4 & b_1 \\
0 & 14 & 10 & b_2 + 4b_1 \\
0 & 0 & 0 & b_1 - \frac{1}{2}b_2 + b_3 \\
\end{bmatrix} \]

Conclusion:

- There is a solution only if \( b_1 - \frac{1}{2}b_2 + b_3 = 0 \)
- This is due to the fact that we have a pivot on 2 rows only
- The columns of \( A \) don’t span \( \mathbb{R}^3 \)
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Homogeneous systems

For a $m \times n$ matrix $A$, consider the system

$$A x = 0.$$ 

Then we have

1. The system is always consistent
   $\iff$ with trivial solution $x = 0$.

2. The system has a nontrivial solution
   $\iff$ the equation has at least one free variable
   $\iff$ $\exists$ a row with no pivot in row-echelon form of $A$
Example of homogeneous system

Matrix:

\[
A = \begin{bmatrix}
0 & 2 & 3 \\
0 & 1 & -1 \\
0 & 3 & 7
\end{bmatrix}
\]

System:

\[
A\mathbf{x} = \mathbf{0}.
\]

Augmented matrix:

\[
A^\# = \begin{bmatrix}
0 & 2 & 3 & 0 \\
0 & 1 & -1 & 0 \\
0 & 3 & 7 & 0
\end{bmatrix}
\]
Example of homogeneous system (2)

Reduced echelon form of $A^\#$:

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Number of solutions: No pivot in 3rd row
$\rightarrow$ infinite number of solutions

Set of solutions:

\[S = \{(s, 0, 0); \ s \in \mathbb{R}\} .\]
2nd example of homogeneous system

Matrix:

\[
A = \begin{bmatrix}
3 & 5 & -4 \\
-3 & -2 & 4 \\
6 & 1 & -8 \\
\end{bmatrix}
\]

System:

\[
A \mathbf{x} = \mathbf{0}.
\]

Augmented matrix:

\[
A^\# = \begin{bmatrix}
3 & 5 & -4 & 0 \\
-3 & -2 & 4 & 0 \\
6 & 1 & -8 & 0 \\
\end{bmatrix}
\]
2nd example of homogeneous system (2)

Reduced echelon form of $A^\#$:

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Number of solutions: No pivot in 3rd row
$\iff$ infinite number of solutions

Set of solutions: for $t \in \mathbb{R}$, of the form

$$x = t \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$
Inhomogeneous systems

**Theorem 11.**

Assume

- Equation $Ax = b$ is consistent
- $p$ is a particular solution of $Ax = b$

Then the general solution of $Ax = b$ can be written as

$$w = p + v_h,$$

where $v_h$ is the general solution of $Ax = 0$
General solution for inhomogeneous systems

Solution of \( Ax = b \) vs solution of \( Ax = 0 \):
Example of inhomogeneous system

Matrices:

\[ A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix} \]

System:

\[ A x = b. \]

Augmented matrix:

\[ A^\# = \begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \]
Example of inhomogeneous system (2)

Reduced echelon form of $A^\#$:

$$
\begin{bmatrix}
1 & 0 & -\frac{4}{3} & -1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

Number of solutions:
No pivot in 3rd row, no equation of the form $0 = 1$
$\rightarrow$ infinite number of solutions

Set of solutions: of the form

$$
x = \begin{bmatrix}
-1 \\
2 \\
0
\end{bmatrix} + t \begin{bmatrix}
\frac{4}{3} \\
0 \\
1
\end{bmatrix}
$$
Outline

1. Systems of linear equations
2. Row reduction and echelon form
3. Vector equations
4. Matrix equation $A \mathbf{x} = \mathbf{b}$
5. Solution sets of linear systems
6. Linear dependence
7. Introduction to linear transformations
8. The matrix of a linear transformation
Definition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be a family of vectors in $\mathbb{R}^n$. Then

1. If there exist $c_1, \ldots, c_p$ not all zero such that

   $$c_1 \mathbf{v}_1 + \cdots + c_p \mathbf{v}_p = 0,$$

   we say that $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent

2. If we have

   $$c_1 \mathbf{v}_1 + \cdots + c_p \mathbf{v}_p = 0 \implies c_1 = c_2 = \cdots = c_p = 0,$$

   we say that $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent
Examples

Simple examples:

- The family \( \{v\} \) is linearly dependent iff \( v = 0 \)
- The family \( \{v_1, v_2\} \) is linearly dependent iff \( v_2 = c v_1 \)
- If \( 0 \) is an element of \( \{v_1, \ldots, v_p\} \)
  \( \leftrightarrow \) then \( v_1, \ldots, v_p \) are linearly dependent
- If \( p > n \) and \( v_1, \ldots, v_p \in \mathbb{R}^n \)
  \( \leftrightarrow \) then \( v_1, \ldots, v_p \) are linearly dependent
Example in $\mathbb{R}^3$

Family of vectors: We consider

$$v_1 = (1, 2, -1), \quad v_2 = (2, -1, 1), \quad v_3 = (8, 1, 1)$$

Then $v_1, v_2, v_3$ are linearly dependent

Proof: The system

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

can be written as:

$$
\begin{align*}
    c_1 + 2c_2 + 8c_3 &= 0 \\
    2c_1 - c_2 + c_3 &= 0 \\
    -c_1 + c_2 + c_3 &= 0
\end{align*}
$$
Example in $\mathbb{R}^3$ (2)

Proof (ctd): System in row-echelon form

$$\begin{bmatrix}
1 & 2 & 8 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

One row is $0$, so that we have linear dependence

Explicit linear dependence: We solve for $c$

$$c_3 = t, \quad c_2 = -3t, \quad c_1 = -2t$$

Then choosing (arbitrary choice) $t = 1$ we get

$$-2v_1 - 3v_2 + v_3 = 0$$
Let $v_1, \ldots, v_p$ be a family of vectors in $\mathbb{R}^n$.

$A = [v_1, \ldots, v_p]$ defined as a $M_{n,p}(\mathbb{R})$ matrix

Then

The vectors $v_1, \ldots, v_p$ are linearly dependent iff the system $Ax = 0$ has a nontrivial solution
Example of application

Family of vectors: We consider

\[ \mathbf{v}_1 = (1, 2, -1), \quad \mathbf{v}_2 = (2, -1, 1), \quad \mathbf{v}_3 = (8, 1, 1) \]

Recall: We have obtained a row-echelon form

\[
\begin{bmatrix}
1 & 2 & 8 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

One row is \( \mathbf{0} \), so that the system \( A\mathbf{x} = \mathbf{0} \) has a nontrivial solution

Conclusion:
\( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) are linearly dependent
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Definition of mapping

Let $n$ and $m$ two integers

Then:

1. A mapping is a rule which assigns to each vector $\mathbf{v} \in \mathbb{R}^n$ a vector $\mathbf{w} = T(\mathbf{v}) \in \mathbb{R}^m$

2. Notation: $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$
Linear transformation

**Definition 15.**

Let

- $n$ and $m$ two integers
- A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$

Then $T$ is a **linear transformation** if it satisfies:

1. $T(u + v) = T(u) + T(v)$
2. $T(c \cdot v) = c \cdot T(v)$
Domain and codomain

**Vocabulary:** For $T : \mathbb{R}^n \to \mathbb{R}^m$ linear

- $\mathbb{R}^n$ is called domain of $T$
- $\mathbb{R}^m$ is called codomain of $T$
- $T(\mathbb{R}^n) \subset \mathbb{R}^m$ is called range of $T$
Examples of linear transformations

**Projection:** Domain $= \mathbb{R}^3$, Codomain $= \mathbb{R}^3$, Range $= \text{Plane}$

**Rotation:** Domain $= \mathbb{R}^2$, Codomain $= \mathbb{R}^2$, Range $= \mathbb{R}^2$
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Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and $(e_1, \ldots, e_n)$ canonical basis of $\mathbb{R}^n$.

Then $T$ is described by the matrix transformation

$$T(x) = Ax,$$

where $A$ is the matrix defined by

$$A = [T(e_1), T(e_2), \ldots, T(e_n)].$$
Example of matrix form

**Linear transformation:** Defined as $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that

$$T(e_1) = T(1, 0) = (2, 3, -1), \quad T(e_2) = T(0, 1) = (5, -4, 7)$$

**Matrix form:** $T(x) = Ax$ with

$$A = [T(e_1), T(e_2)] = \begin{bmatrix} 2 & 5 \\ 3 & -4 \\ -1 & 7 \end{bmatrix}$$

**General expression:**

$$T(x) = Ax = \begin{bmatrix} 2x_1 + 5x_2 \\ 3x_1 - 4x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$
Examples of transformations in $\mathbb{R}^2$

Reflection on $x_1$-axis:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
Examples of transformations in $\mathbb{R}^2$ (2)

Reflection on line $x_2 = x_1$:

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
Examples of transformations in $\mathbb{R}^2$ (3)

Rotation:

$$A_3 = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$
Composition of transformations in $\mathbb{R}^2$

Reflection on $x_1$ then reflection on $x_2 = x_1$:

$$A = A_2 A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Resulting transformation:

Rotation with $\varphi = \frac{\pi}{2}$
Properties of $T$ according to its matrix

**Theorem 17.**

Let

- $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear transformation
- $A$ its standard matrix form

Then

1. $T$ is onto
   - iff columns of $A$ span $\mathbb{R}^m$
   - iff there is a pivot in every row of $A$ (Thm 9)

2. $T$ is one-to-one
   - iff columns of $A$ are linearly independent
   - iff $\mathbf{0}$ is the unique solution of $Ax = \mathbf{0}$ (Prop 13)
Example from $\mathbb{R}^2$ to $\mathbb{R}^3$

Transformation: For $\mathbf{x} \in \mathbb{R}^2$

$$T(\mathbf{x}) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Matrix:

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$$

Properties of $T$:

1. Only two columns in $A \implies$ is not onto in $\mathbb{R}^3$
2. Columns of $A$ are independent $\implies T$ is one-to-one