MA 262

Lecture 1 Sections 1.1 and 1.2

Prof. Tindel will be back on thursday.

I am Prof. Sa Barreto

Introduction To Differential Equations

Goals of the Course.

1) Learn about differential Equations.

2) Systems of Diff. Eq.

3) Linear Algebra.

Examples of Differential Equations

1) Simplest example: $\dfrac{dy}{dx} = f(x)$

$$y(x) = \int f(x)\,dx + C$$

Example: $\dfrac{dy}{dx} = \dfrac{3x}{1+x^2}$; $y(x) = \displaystyle\int \frac{3x}{1+x^2}\,dx + C$

$$\int \frac{3x}{1+x^2}\,dx = \int \frac{3}{u}\,\frac{du}{2} = \frac{3}{2}\int \frac{du}{u} = \frac{3}{2}\ln|u|.$$

$$= \frac{3}{2}\ln(1+x^2)$$

$u = 1+x^2$, $du = 2x\,dx$

$x\,dx = \dfrac{du}{2}$

$$\boxed{y(x) = \frac{3}{2}\ln(1+x^2) + C}$$

this is called the general solution because C is not specified.

But if we add a condition.

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{3x}{1+x^2} \\ y(0) = 5 \end{cases}$$

$$y(x) = \frac{3}{2}\ln(1+x^2) + C$$

$$y(0) = \frac{3}{2}\ln(1) + C = C$$

$$C = 5$$

$$\boxed{y(x) = \frac{3}{2}\ln(1+x^2) + 5}$$

A particular solution.

Example: 2 $\qquad \frac{dy}{dx} = x\,y \qquad\qquad y(x) = \int x\, y(x)\, dx + C.$

Idea: To transform this into an equation as in the 1st example.

1. $y(x) = 0$ Solves the equation

2. $y(x) \neq 0 \qquad \frac{1}{y}\frac{dy}{dx} = x \quad ; \quad \frac{d}{dx} \ln|y(x)| = x$

Chain Rule: $\quad \frac{d}{dx} \ln|y(x)| = \frac{1}{y(x)} \cdot \frac{dy}{dx}$

So we have: $\quad \frac{d}{dx} \ln|y(x)| = x.$

Integrate this equation:

$$\int \frac{d}{dx} \ln |y(x)| \, dx = \int x \, dx + c$$

$$\ln |y(x)| = \frac{1}{2}x^2 + c$$

$$|y(x)| = e^{\frac{1}{2}x^2 + c} = e^c \cdot e^{\frac{1}{2}x^2}$$

$$y(x) = e^c \cdot e^{\frac{1}{2}x^2} \quad \text{or} \quad y(x) = -e^c e^{\frac{1}{2}x^2}.$$

$$y(0) = e^c \qquad\qquad y(0) = -e^c$$

$$\boxed{y(x) = y(0)\, e^{\frac{1}{2}x^2}}$$

Examples of where such equations appear.

1) Population growth.

$P(t)$ = Population of a certain species at time t.

$$\frac{dP(t)}{dt} = K\,P(t) \qquad K > 0$$

$$\frac{1}{P(t)}\frac{dP}{dt} = K \; ; \; \frac{d}{dt}\ln P(t) = K$$

$$\ln P(t) = \int K\,dt + C = Kt + C.$$

$$P(t) = e^{Kt+C} = e^{C}.e^{Kt}.$$

$$P(0) = C \qquad \boxed{P(t) = P(0)\,e^{Kt}}$$

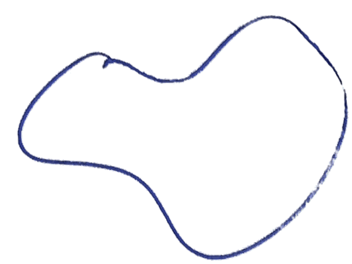
Radioactive decay:

$Q(t)$ = Amount of a radioactive Substance at time t.

$$\frac{dQ}{dt} = -k\,Q(t), \qquad k>0$$

$$\boxed{Q(t) = Q(0)\,e^{-kt}}$$

Newton's Law of Cooling:



θ

$T(t)$ = temperature of θ at time t

A = temperature of the room = Constant

$$\frac{dT}{dt} = -k(T-A), \qquad k>0.$$

$$\frac{1}{T-A}\,\frac{dT}{dt} = -K$$

$$\frac{d}{dt}\ln|T-A| = -K \quad \ldots \text{ Exercise}$$

Equations of Higher order

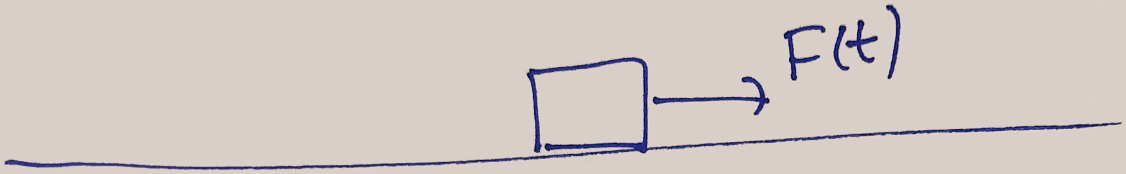
$$y''(x) + 5y'(x) + 10y = 0 \quad \text{is of order } 2$$

$$y'' = \frac{d^2y}{dx^2}, \quad y' = \frac{dy}{dx}$$

$$y^{(3)}(x) + 10\,y''(x) + e^{x} y(x) = \sin x$$

$$y^{(3)} = \frac{d^3y}{dx^3} = \text{3rd derivative}$$

Newton's Law of motion.



$$F(t) = m\,a(t).$$

$x(t)$ = position of the object at time t

$x'(t)$ = velocity $\qquad$ $x''(t)$ = acceleration.

$$\boxed{F(t) = m\,x''(t)}$$

Example: $F(t) = \cos t$.

$$m\, x''(t) = \cos t.$$

$$x''(t) = \frac{1}{m}\cos t.$$

$$x'(t) = \frac{1}{m}\int \cos t \, dt + c = \frac{1}{m}\sin t + c.$$

$$\boxed{x(t) = -\frac{1}{m}\cos t + ct + c_1}$$

General Solution.

Example: $F(t) = \cos t - 10\, x'(t)$.

$$\boxed{m\, x''(t) = \cos t - 10\, x'(t)}$$

Example $y''(x) - 2y'(x) + y(x) = 0$.

Look for solution $y(x) = e^{rx}$, r constant

$y(x) = e^{rx}$, $y'(x) = r e^{rx}$, $y''(x) = r^2 e^{rx}$

$$r^2 e^{rx} - 2r e^{rx} + e^{rx} = 0$$

$$0 \neq e^{rx}(r^2 - 2r + 1) = 0$$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

<u>Conclusion:</u> $y(x) = e^x$ solves the equation.

Check that $y(x) = xe^x$ also solves the equation

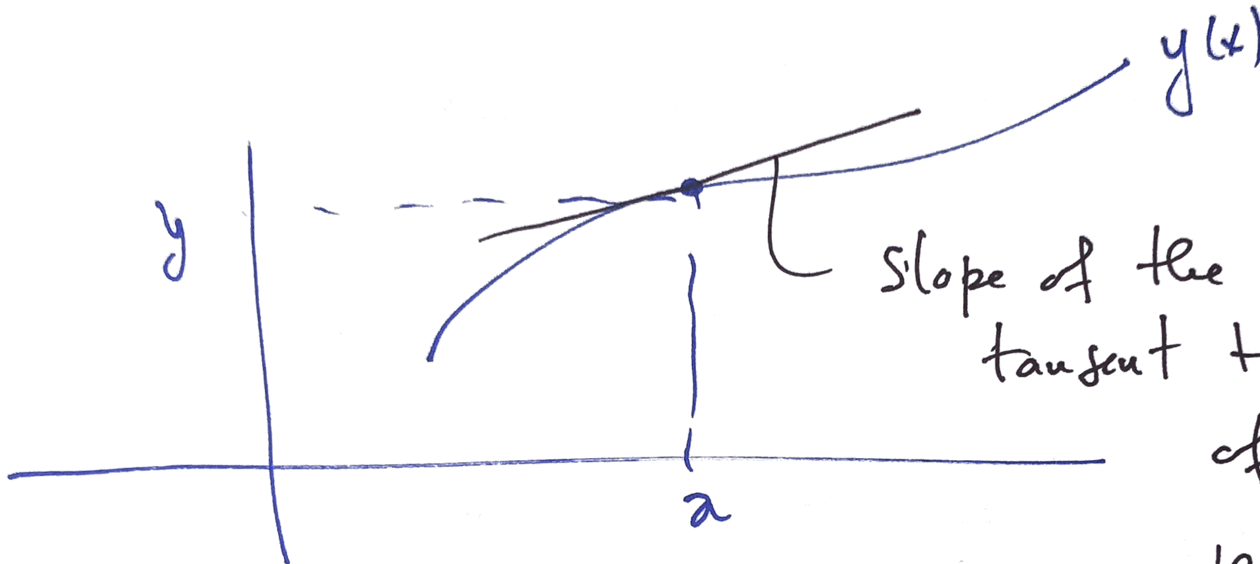
$$y(x) = xe^x \quad , \quad y'(x) = e^x + xe^x \quad ;$$

$$y''(x) = e^x + e^x + xe^x = 2e^x + xe^x$$

Substitute this back into the equation

$$y''(x) - 2y'(x) + y(x) = 0$$

Geometric Interpretation of Equations of order 1.

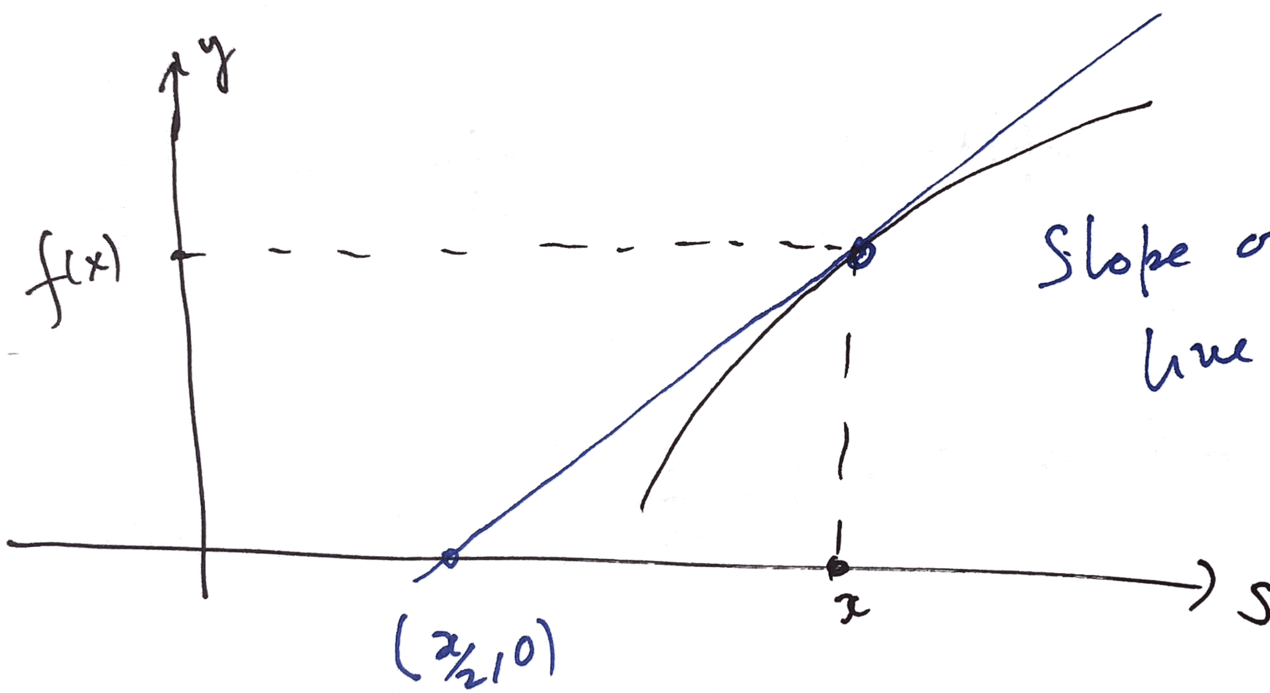


Slope of the line tangent to the graph of $y(x)$ at the point (x, y) $= y'(x)$.

An equation

$$y'(x) = F(x, y)$$

Example: Find a function $f(x)$ such that the line tangent to the graph of $f(x)$ at $(x, f(x))$ intersects the x-axis at $(x/2, 0)$



$$y - f(x) = f'(x)(s - x)$$

when $y = 0$, $s = x/2$

$$0 - f(x) = f'(x)(x/2 - x)$$

$$-f(x) = f'(x)\left(-\tfrac{x}{2}\right)$$

$$\boxed{f(x) = f'(x)\,\tfrac{x}{2}}$$

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)} = \frac{2}{x} \qquad \frac{d}{dx}\ln|f(x)| = \frac{2}{x}$$

$$\ln|f(x)| = \int \frac{2}{x}\,dx + c = 2\ln|x| + c$$

$$\ln|f(x)| = \ln x^2 + c$$

$$|f(x)| = e^{c+\ln x^2} = e^c \cdot e^{\ln x^2}$$

$$|f(x)| = x^2 e^c$$

$$f(x) = \pm e^c x^2$$

$$\boxed{f(x) = k x^2}$$