# EQUILIBRIUM SOLUTIONS

General form of 1st order eq

$$y' = f(x, y)$$

Autonomous eq

$$y' = f(y)$$

Note : eq can be written as

$$\frac{dy}{dx} = f(y)$$

$$\iff \frac{dy}{f(y)} = dx \rightarrow \text{separable}$$

Today, we are not going to solve the eq.
Instead, we are going to analyze equilibrium.

was called c last week

# Verhulst eq

$$y' = \underbrace{r\left(1 - \frac{y}{K}\right)y}_{f(y)}$$

## Plot of f

$f$ is quadratic in $y$

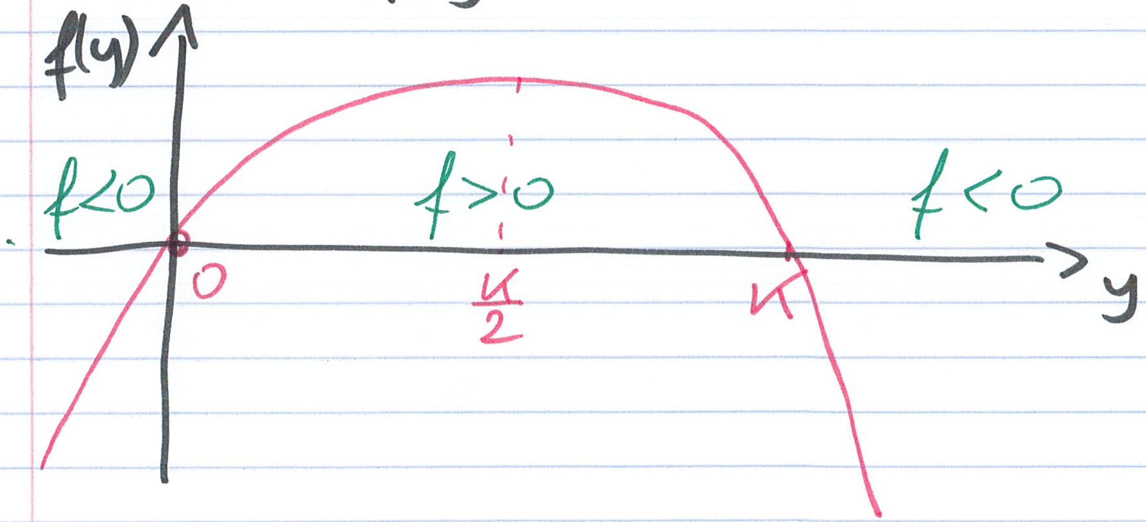
Its graph is a parabola

In addition, $f(y)$ is $0$ if

$$y = 0 \quad \text{or} \quad y = K$$

The quadratic term is $-\frac{y^2}{K} r$

↳ "unhappy" parabola



$y' = f(y)$ , $f(y) = r\left(1-\frac{y}{K}\right)y$

③

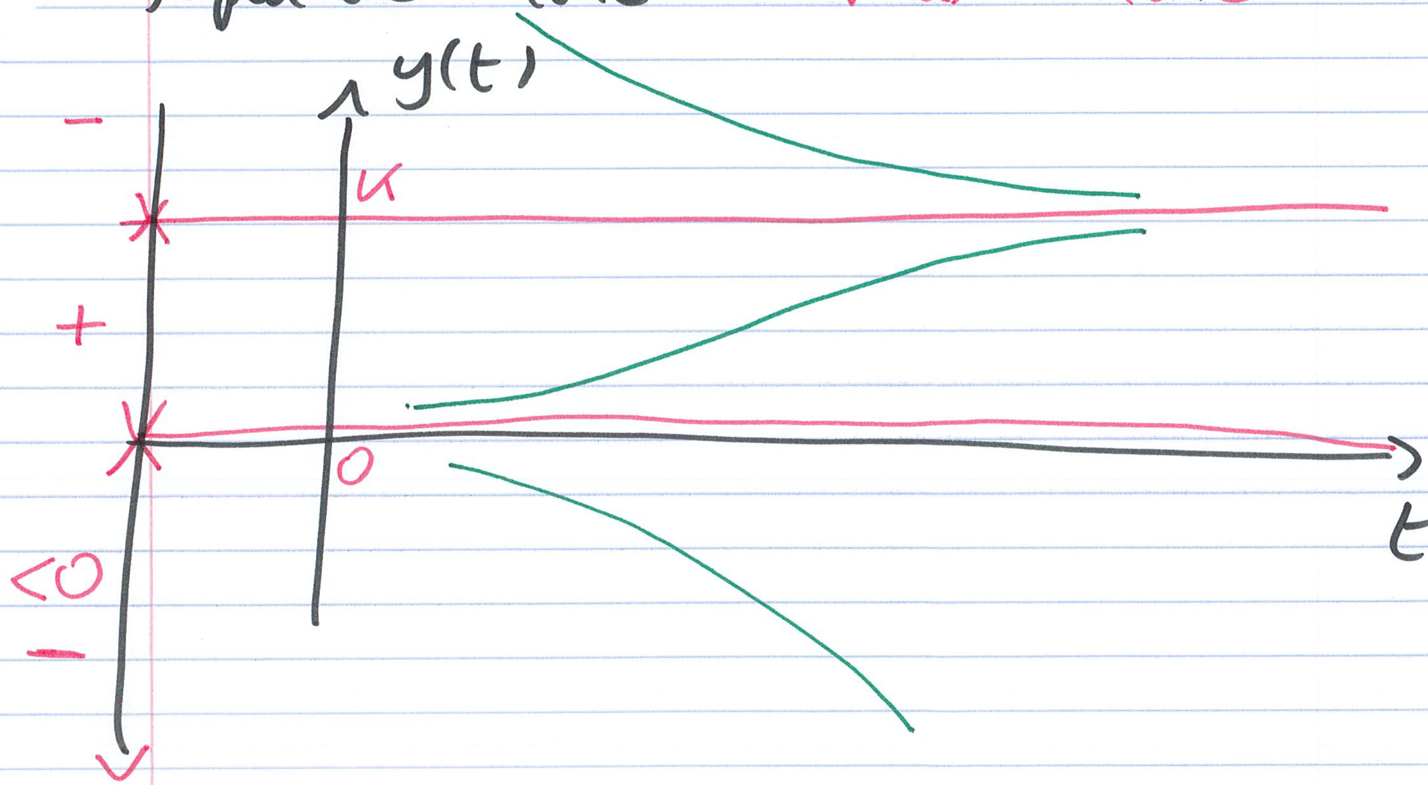
Summary so far

$f < 0$ if $y \in (-\infty, 0)$

$f > 0$ if $y \in (0, K)$

$f < 0$ if $y \in (K, \infty)$

Next step: record the information about the sign of $f$ on a separate line → phase line




↳ phase line

Equilibrium Here we have two equilibrium, characterized by $f(y)=0$:

$$y=0 \quad \text{and} \quad y=K$$

Those two equilibrium are different in nature

(i) For $y=0$, if we start from an initial condition which is $y_0<0$ or $y_0 \in (0,K)$, then

- If $y_0<0$, $\lim_{t\to\infty} y(t) = -\infty$
- If $y_0 \in (0,K)$, $\lim_{t\to\infty} y(t) = K$

$\neq 0$

As an equilibrium, 0 is $\boxed{\text{unstable}}$

(ii) For $y = K$, if we start from $y_0 \in (0, K)$ or $y_0 > K$ we have

$$\lim_{t \to \infty} y(t) = K$$

As an equilibrium, $K$ is $\boxed{\text{stable}}$

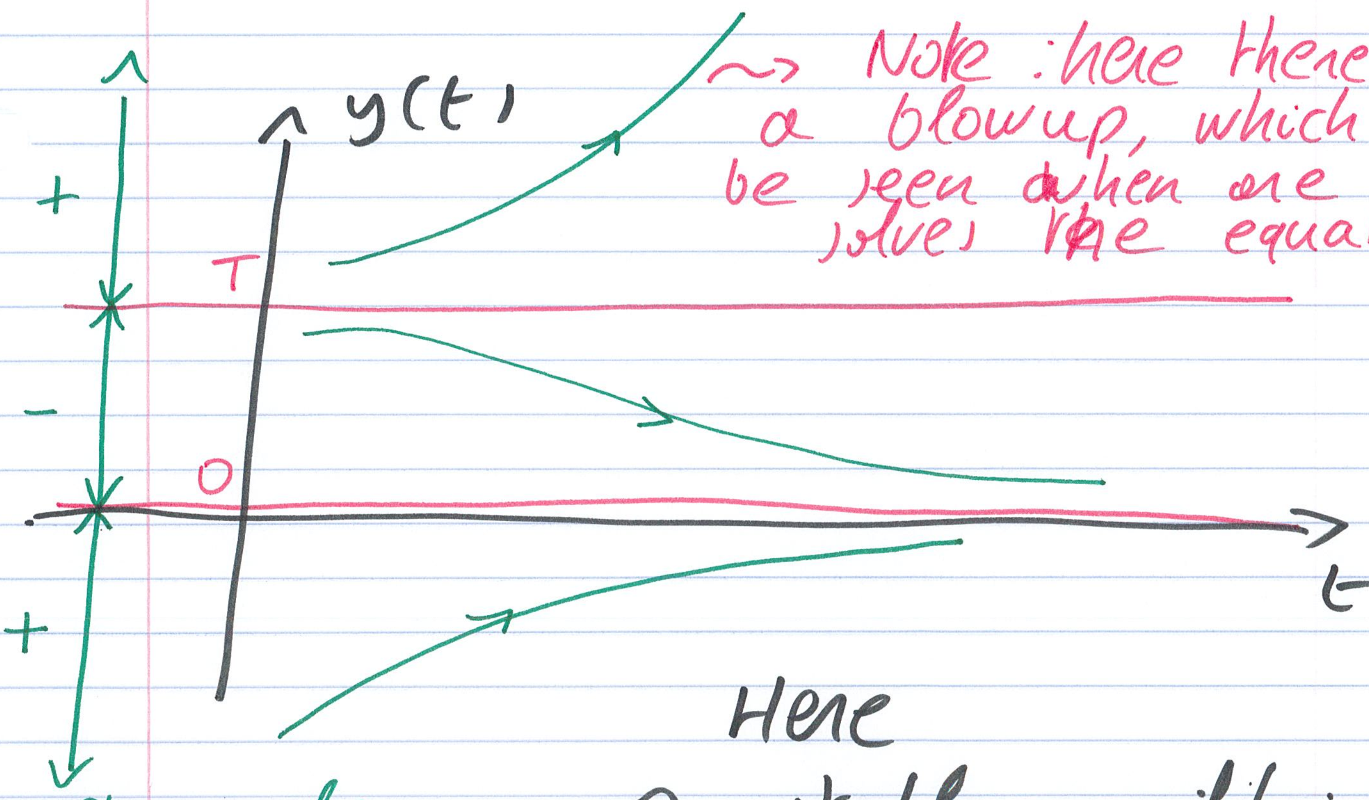
Rmk For an eq. of the form $y' = f(y)$, the solutions to $f(y) = 0$ are called equilibrium or critical points

# Critical threshold eq

$$\frac{dy}{dt} = \underbrace{-r\left(1-\frac{y}{T}\right)y}_{f(y)}$$

we have $f(y) > 0$ if $y \notin (0,T)$
$f(y) < 0$ if $y \in (0,T)$

Note: here there is a blowup, which can be seen when one solves the equation




Phase line

Here
0 stable equilibrium
T unstable equilibrium

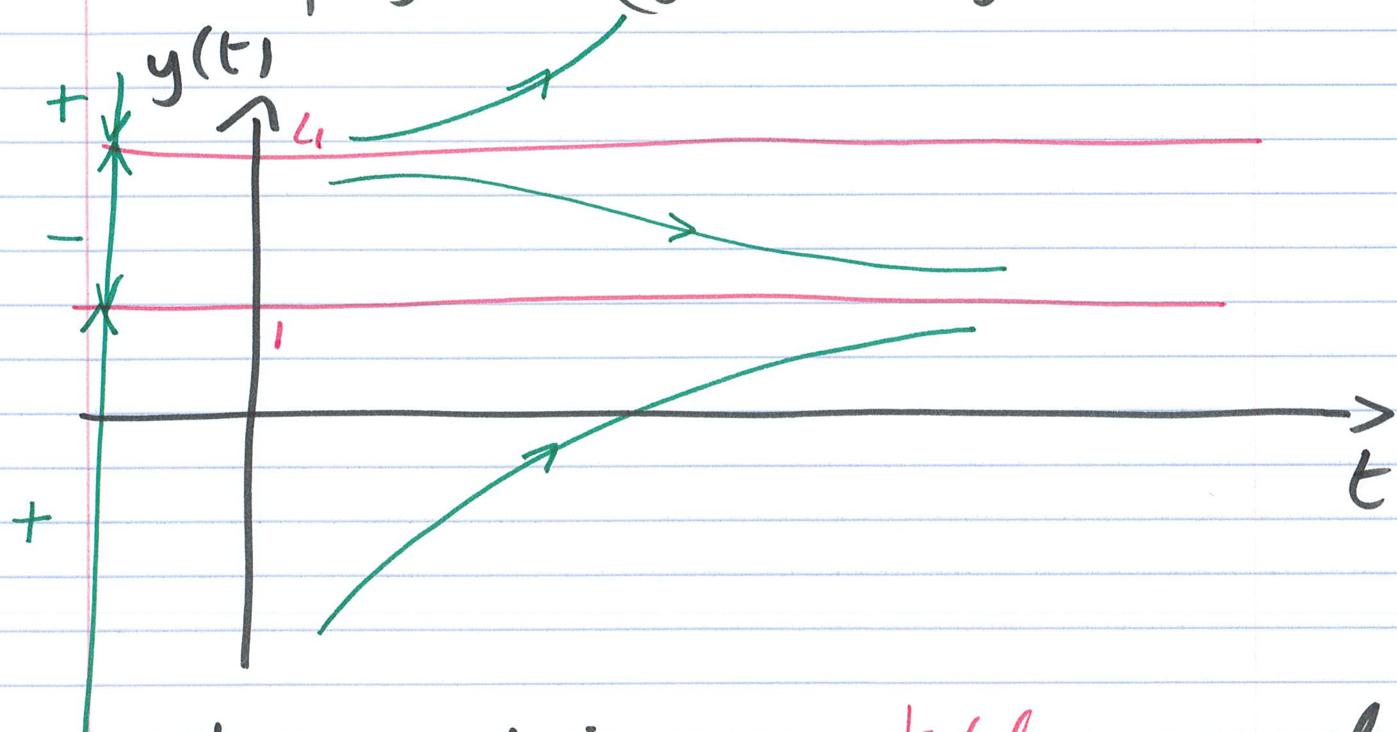
Eq $y' = \underbrace{y^2 - 5y + 4}_{f(y)}$

Roots for $f$

$y = 1$ and $y = 4$

Thus

$$f(y) = (y-1)(y-4)$$

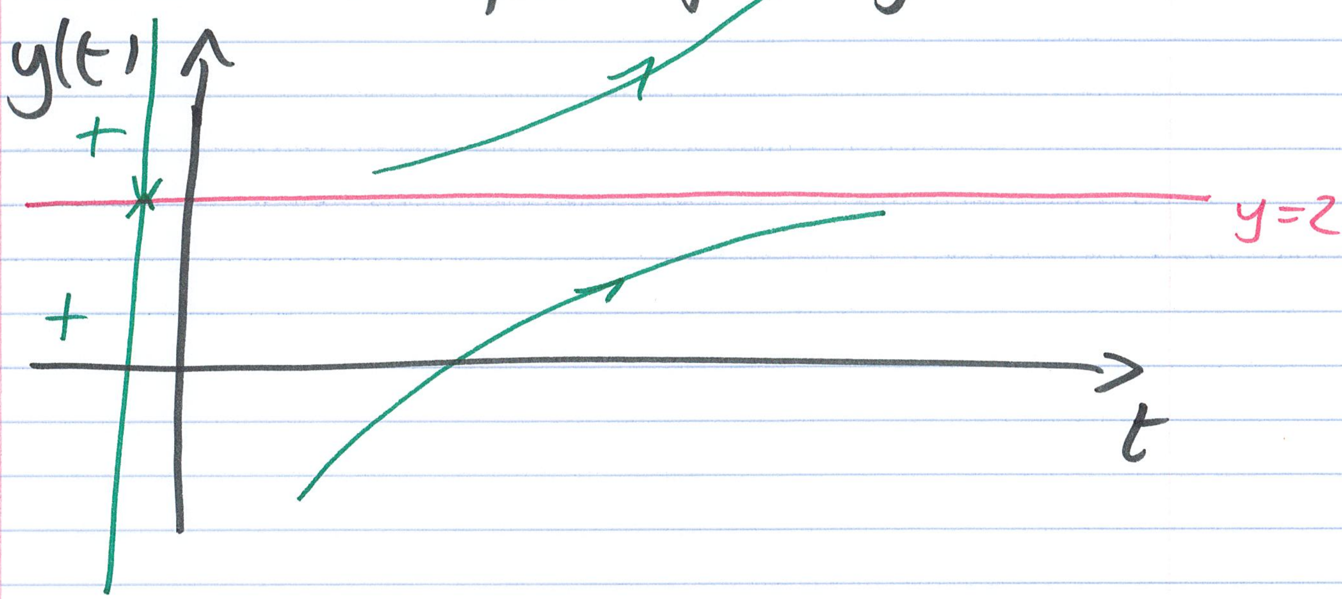



Thus 1 is a stable equil.

4 is an unstable equil.

Eg $\quad y' = \underbrace{(y-2)^2}_{f(y)}$

Double root for $f$: $y=2$




$y=2$ is stable if $y_0 < 2$
$y=2$ is unstable if $y_0 > 2$

We say that $y=2$ is a $\boxed{\text{semi-stable}}$ equilibrium

## Back to tank problems

We look for an eq. for $Q(t)$, where $Q(t)$ = quantity of salt in the tank, at time $t$.

Relation

$$\frac{dQ}{dt} = \text{rate}_{in} - \text{rate}_{out}$$

Then $\text{rate}_{in} = C_{in} \times r_{in}$

$$= \frac{1}{4} r = \frac{r}{4}$$

$\text{rate}_{out} = C_{out} \times r_{out}$

$$= \frac{Q(t)\,r}{V} = \frac{Q}{100}\, r$$

Note: $V$ is constant, since

$$r_{in} = r_{out} = r$$

Eq

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{Q}{100} r$$

$$\Leftrightarrow \frac{dQ}{dt} + \frac{r}{100} Q = \frac{r}{4}$$

We get a linear eq, with

$$\mu = e^{\frac{r}{100} t}$$