Reduced Row-echelon

Example of system

with 4 unknown, 3 eq only

↳ we cannot expect to have 1 unique solution

either ∞ or 0 solutions

1st pivot

$$A^{\#} = \begin{pmatrix} \boxed{1} & -2 & 2 & -1 & 3 \\ 3 & 1 & 6 & 11 & 16 \\ 2 & -1 & 4 & 4 & 9 \end{pmatrix}$$

$A_{13}(-2)$

$A_{12}(-3)$

2nd pivot

$$\sim \begin{pmatrix} 1 & -2 & 2 & -1 & 3 \\ 0 & \boxed{7} & 0 & 14 & 7 \\ 0 & 3 & 0 & 6 & 3 \end{pmatrix}$$

$M_3(1/3)$

$M_2(1/7)$

$$\sim \begin{pmatrix} 1 & -2 & 2 & -1 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 & 1 \end{pmatrix}$$

$$\overset{A_{23}(-1)}{\sim} \begin{pmatrix} \boxed{1} & -2 & 2 & -1 & 3 \\ 0 & \boxed{1} & 0 & 2 & 1 \\ 0 & 0 & \underbrace{0 & 0}_{} & 0 \end{pmatrix}$$

triangular shape

row with 0's, at the bottom of the matrix

Here we have · 4 unknown
· 2 leading 1's

Thus the variables (x_3, x_4) which do not correspond to leading 1's will be **free variables**

We set $x_3 = s \in \mathbb{R}$
$x_4 = t \in \mathbb{R}$

Then we use the row-ech. form of $A^\#$ to get

$$x_2 = 1 - 2x_4 = 1 - 2t$$

$$\begin{aligned} x_1 &= 2x_2 - 2x_3 + x_4 + 3 \\ &= \cancel{1}2 - \cancel{2}4t - 2s + t + 3 \\ &= \cancel{4}5 - 2s - 3t \end{aligned}$$

Solution set

$$S = \{(x_1, x_2, x_3, x_4) = (5-2s-3t, 1-2t, s, t); s,t \in \mathbb{R}\}$$

$$= \{(5,1,0,0) + s(-2,0,1,0) + t(-3,-2,0,1); s,t \in \mathbb{R}\}$$

↳ plane in $\mathbb{R}^4$

Example of reduced row-echelon

1st pivot $\qquad$ 2nd pivot

$$A=\begin{pmatrix}\boxed{1} & 9 & 26\\ 0 & \boxed{14} & 28\\ 0 & -14 & -28\end{pmatrix}$$

Already good fr reduced row-echelon

$$\overset{M_2(1/14)}{\sim}\begin{pmatrix}1 & 9 & 26\\ 0 & 1 & 2\\ 0 & -14 & -28\end{pmatrix}$$

$$\overset{A_{21}(-9)}{\sim}\begin{pmatrix}1 & 0 & 8\\ 0 & 1 & 2\\ 0 & -14 & -28\end{pmatrix}$$

$$\overset{A_{23}(14)}{\sim}\begin{pmatrix}1 & 0 & 8\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$$

Reduced row-ech.

Row-echelon form of a system

If

$$A^{\#} = \begin{pmatrix} 3 & -2 & 2 & 4 \\ 1 & -2 & 1 & 5 \\ 2 & -1 & -2 & -1 \end{pmatrix}$$

$$\vdots \overset{P_{12}}{\sim} \cdots$$

$$\sim \begin{pmatrix} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Then one can solve the system:

$$x_3 = 2$$

$$x_2 = 5 - 3x_3 = 5 - 6 = -1$$

$$x_1 = 2x_2 - x_3 + 5$$

$$= -2 - 2 + 5 = 1$$

Unique solution:

$$x = (x_1, x_2, x_3) = (1, -1, 2)$$

Reduced row echelon form

On ~~the same~~ a different system

$$A^{\#} = \xcancel{\begin{pmatrix} 3 & -2 & 2 & 9 \\ 1 & -2 & 1 & 5 \\ 2 & -1 & -2 & -1 \end{pmatrix}} \quad \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \end{pmatrix}$$

$\overset{\cdot P_{12}}{\sim}$ .... (longer computation)

$$\sim \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Now solve the system:

$$\left.\begin{aligned} x_3 &= 2 \\ x_2 &= -1 \\ x_1 &= 5 \end{aligned}\right)$$

we get a unique solution immediately

Pros and cons of reduced row-echelon

Pros: Once we have a reduced row-echelon, system is trivially solved

Cons: It is (much) longer to get the reduced row-echelon form

Homogeneous system

whenever $b = 0$.
In compact form, it can be written as

$$A\vec{x} = \vec{0} \rightarrow \text{column with } 0\text{'s}$$

Rmk The vector

$$\vec{x} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is always a solution to the system → the system is always consistent

we either have

- 1 solution, which is $\vec{0}$
- ∞ number of solutions

Identity matrix

$$I_n = \begin{pmatrix} 1 & & & \\ & 1 & & \text{\Large 0} \\ & & \ddots & \\ \text{\Large 0} & & & 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example of hom. system

start from the matrix

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 0 & \boxed{1} & -1 \\ 0 & 3 & 7 \end{pmatrix}$$

$$\overset{\substack{A_{23}(-3) \\ A_{21}(-2)}}{\sim} \quad \cdots$$

$$\sim \begin{pmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix} \neq I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Conclusion: ∞ number of solution for the system

$Ax = 0$

⑪

we can solve the system

x_1 is set as a free var.

$x_1 = s \in \mathbb{R}$

Then $x_2 = 0$

$x_3 = 0$

Solution set for $Ax = 0$:

$S = \{(s, 0, 0); s \in \mathbb{R}\}$

Example of matrix

$$A = \begin{pmatrix} 3/2 & 2/3 & 1/5 \\ 0 & 5/4 & -3/7 \end{pmatrix}$$

$$\hookrightarrow A^T = \begin{pmatrix} 3/2 & 0 \\ 2/3 & 5/4 \\ 1/5 & -3/7 \end{pmatrix}$$

Coefficients

$$a_{\underset{\text{1st row}}{\textcircled{1}}\underset{\text{2nd column}}{\textcircled{2}}} = 2/3$$

$$a_{23} = -3/7$$

vectors

$$a = \left(2/3, -1/5 \quad 4/7\right)$$

is can be seen as a

1 × 3 matrix