Determinants

Examples of determinants

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$$

$$= 1 \times 5 - 2 \times 3 = -1$$

$$A = \begin{pmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{pmatrix}$$

$$\det(A) = \left| \begin{matrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{matrix} \right| \begin{matrix} 1 & 3 \\ 2 & 5 \\ 1 & 0 \end{matrix}$$

$$= 30 + (-3) + 0$$
$$- (-20 + 0 + \cancel{18}\ 36)$$

$$= 27 - 16 = 11$$

Example of minor

a_{12}

$$A = \begin{pmatrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{pmatrix}$$

$$A_{12} = \begin{vmatrix} 2 & -1 \\ 1 & 6 \end{vmatrix}$$

$$= 12 + 1$$

$$= 13$$

Related cofactor

$$C_{12} = (-1)^{1+2} A_{\cancel{4}}^{12}$$

$$= -13$$

Rule for signs

$$A = \begin{pmatrix} 1_{+} & 3_{-} & -4_{+} \\ 2_{-} & 5_{+} & -1_{\ominus} \\ 1_{+} & 0_{-} & 6_{+} \end{pmatrix}$$

Alternate signs assignment

Rmk In order to use Thm 17 in an efficient way, one has to look for rows/columns with 0's

Application of Thm 17

$$\begin{vmatrix} 1^{+} & 3 & -4 \\ 2^{-} & 5 & -1 \\ 1^{+} & 0^{-} & 6^{+} \end{vmatrix}$$

Expand wrt 3rd row

$$= 1 \times \begin{vmatrix} 3 & -4 \\ 5 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 1 & -4 \\ 2 & -1 \end{vmatrix}$$

$$+ 6 \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$$

$$= 17 + 6(-1)$$

$$= 11$$

(same result as previous computation)

Determinant for an upper triag. matr

$$A = \begin{pmatrix} 1 & 3 & -4 \\ 0 & 5 & -1 \\ 0 & 0 & 6 \end{pmatrix}$$

Then

$$\det(A) = a_{11} \times a_{22} \times a_{33}$$
$$= 1 \times 5 \times 6 = 30$$

Next step : reduce the general case to the det of a triangular matrix

↳ row echelon form

Reducing det. to upper triangular mat

$$\begin{vmatrix} \boxed{1} & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{vmatrix}$$

$A_{13}(-1)$
$A_{12}(-2)$

$$= \begin{vmatrix} 1 & 3 & -4 \\ 0 & -1 & 7 \\ 0 & -3 & 10 \end{vmatrix}$$

$M_3(-1)$
$M_2(-1)$

$$= + \begin{vmatrix} 1 & 3 & -4 \\ 0 & 1 & -7 \\ 0 & 3 & -10 \end{vmatrix}$$

$A_{23}(-3)$

$$= \begin{vmatrix} 1 & 3 & -4 \\ 0 & 1 & -7 \\ 0 & 0 & 11 \end{vmatrix}$$

$= 11$ (same value as before)

Applying Cramer's rule

$Ax = b$ with

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 1 & -5 \\ -2 & -1 & 4 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$$

Aim: solve for x_1. Then

$$A_1(b) = \begin{pmatrix} 4 & 2 & -1 \\ -3 & 1 & -5 \\ 0 & -1 & 4 \end{pmatrix}$$

$$x_1 = \frac{\det(A_1(b))}{\det(A)} = \frac{17}{8}$$

Rmk Thm 19 is worth using if we are asked to solve for 1 coordinate (e.g x_2)

Inverting with determinants

$$A = \begin{pmatrix} 2^{+} & 0^{-} & -3^{+} \\ -1^{-} & 5^{+} & 4^{-} \\ 3^{+} & -2^{-} & 0^{+} \end{pmatrix}$$

Then $C_{11} = + \begin{vmatrix} 5 & 4 \\ -2 & 0 \end{vmatrix} = 8$

$C_{12} = - \begin{vmatrix} -1 & 4 \\ 3 & 0 \end{vmatrix} =$

$= -(-12) = 12 \ldots$

$$M_c = \begin{pmatrix} 8 & 12 & -13 \\ 6 & 9 & 4 \\ 15 & -5 & 10 \end{pmatrix}$$

$$\operatorname{adj}(A) = M_c^T = \begin{pmatrix} 8 & 6 & 15 \\ 12 & 9 & -5 \\ -13 & 4 & 10 \end{pmatrix}$$

$\det(A) = 55$

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)}$$

$$= \frac{1}{55}\begin{pmatrix} 8 & 6 & 15 \\ 12 & 9 & -5 \\ -13 & 4 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 8/55 & 6/55 & 15/55 \\ & \vdots & \end{pmatrix}$$

Rmk (i) Gauss-Jordan is more efficient to compute A^{-1}

(ii) We might want to use det if we are asked to compute

$$(A^{-1})_{13} = \frac{\mathrm{adj}(A)_{13}}{\det(A)} = \frac{C_{31}}{\det(A)}$$