Linear Combinations

$M_2(\mathbb{R})$ as a vector space

For $A, B \in M_2(\mathbb{R})$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Then

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

$$3A = \begin{pmatrix} 3a_{11} & 3a_{12} \\ 3a_{21} & 3a_{22} \end{pmatrix}$$

Then check that $M_2(\mathbb{R})$ equipped with $+, \cdot$ is a vector space (conditions 1-10)

Example H on the slide

Here $V = \mathbb{R}^3$, $H \subset \mathbb{R}^3$ subset

$$\boxed{H : x_3 = 0}$$

Rmk: Later we will also be able to write

$$H = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Claim: H is a subspace

(1) $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ satisfies $x_3 = 0$

Thus $\vec{0} \in H$

(2)

(2) Take

$$x = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix}$$

Consider $z = x + y$.

Then

$$z = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ 0 \end{pmatrix} \in H$$

(3) Take $x = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$, $\alpha \in \mathbb{R}$

$$z = \alpha x$$

$$= \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ 0 \end{pmatrix} \in H$$

Conclusion: H is a subspace

Hom. linear system in $\mathbb{R}^3$

$$\begin{cases} x_1 + 2x_2 - x_3 = 0 \\ 2x_1 + 5x_2 - 4x_3 = 0 \end{cases}$$

$S =$ sol. set

(1) Take $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then $x \in S$

Rmk: The system is of the form $Ax = 0$. We have seen in the previous chapter that $x = \vec{0}$ is a trivial solution to this kind of system

(2) Take $x, y \in S$.
Define $z = x+y$
Question: $z \in S$?

$$z_1 + 2z_2 - z_3$$
$$= (x_1+y_1) + 2(x_2+y_2) - (x_3+y_3)$$
$$= \underbrace{(x_1 + 2x_2 - x_3)}_{=0 \text{ since } x \in S} + \underbrace{(y_1 + 2y_2 - y_3)}_{=0 \text{ since } y \in S}$$
$$= 0$$

In the same way,

$$2z_1 + 5z_2 - 4z_3 = 0$$

Thus $z \in S$

(3) Take $x \in S$, $\alpha \in \mathbb{R}$

Set $z = \alpha x$

Then (check)

$z_1 + 2z_2 - z_3 = 0$

$2z_1 + 5z_2 - 4z_3 = 0$

Thus $z \in S$

Conclusion: S is a subspace

Method 2: Solve the system

we have

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 5 & -4 \end{pmatrix}$$

$$\sim \begin{pmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & -2 \end{pmatrix}$$

x_3: free var.

$x_3 = s$

Then

$x_2 = 2x_3 = 2s$

$x_1 = -3x_3 = -3s$

Solution set S

$S = \{(-3s, 2s, s) ; s \in \mathbb{R}\}$

$S = \{ s(-3, 2, 1) ; s \in \mathbb{R}\}$

Thus we can check that

$0 \in S$

stability by $+$ and $\cdot$

Rmk S is solution set of a hom. linear system. We will also write

$$S = \text{Span}\left\{\begin{pmatrix}-3\\2\\1\end{pmatrix}\right\}$$

Counter-example $S \subset \mathbb{R}^2$

$$S = \{ (r, -3r+1) \; ; \; r \in \mathbb{R} \}$$

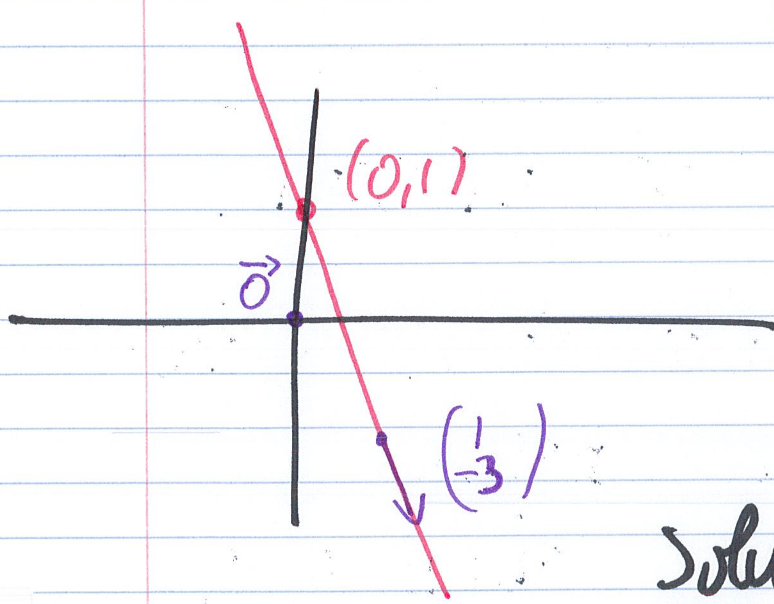
Then

(1) $0 \notin S$

(2) $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in S \quad y = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

If $z = x+y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, then $z \notin S$

(on S, if $z_1 = 1 = r$ then $z_2 = -2 \neq 1$)



Rmk

Solution to hom. syst. are subspaces

Solution to non hom. syst are not subspaces

Example of Null(A)

5 unknowns, 3 equations

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

$$S = \text{Null}(A) = \{x ; Ax = 0\} \subset \mathbb{R}^5$$

Row-echelon form

$$A \sim \begin{pmatrix} \boxed{1} & -2 & 0 & -1 & 3 \\ 0 & 0 & \boxed{1} & 2 & -2 \\ 0 & \boxed{0} & 0 & \boxed{0} & \boxed{0} \end{pmatrix} \cancel{0}$$

Free var: $x_2 = r$, $x_4 = s$, $x_5 = t$

Then $x_3 = -2x_4 + 2x_5$

$= -2s + 2t$

$x_1 = 2x_2 + x_4 - 3x_5$

$= 2r + s - 3t$

Solution set

$$S = \{ (2r + s - 3t, r, -2s + 2t, s, t); \; r, s, t \in \mathbb{R} \}$$

$$= \left\{ r \overbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}^{v_1} + s \overbrace{\begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}}^{v_2} + t \overbrace{\begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}}^{v_3} ; \; s, r, t \in \mathbb{R} \right\}$$

$$= \operatorname{Span} \{ v_1, v_2, v_3 \}$$