Row and col spaces

Basis: set $\{v_1, \dots, v_n\}$ in V s.t.

(i) $\{v_1, \dots, v_n\}$ lin. indep.

(ii) $\text{Span}\{v_1, \dots, v_n\} = V$

Dimension

(i) If $\{v_1, \dots, v_n\}$ is a basis for V, then any basis will have n elements

Def: $n = \text{Dim}(V)$

(ii) Any family $\{u_1, \dots, u_{n+1}\}$ is lin dep.

Dim of $\mathbb{R}^3$. We have seen that if

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

then $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$

There are 3 elements in this basis

$$\overset{\text{Def 16}}{\Longrightarrow} \quad \dim(\mathbb{R}^3) = 3$$

A linearly dep. family.

Consider $\{u_1, u_2, u_3, u_4\}$ with

$$u_1 = \begin{pmatrix} \pi \\ 5 \\ -7 \end{pmatrix}, u_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, u_4 = \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix}$$

$$\overset{\text{Thm 15-1}}{\Longrightarrow} \quad \{u_1, u_2, u_3, u_4\} \text{ lin dep.}$$

Dimension of $\mathbb{P}_2$

$\mathbb{P}_2 = \{ \text{polynomials with deg} \le 2 \}$

$$= \{ \underbrace{a + bt + ct^2}_{=p(t)} ;\ a, b, c \in \mathbb{R} \}$$

Canonical basis for $\mathbb{P}_2$

$a + bt + ct^2$ $\qquad \cancel{0}$ $\quad 1$

A basis is a set of lin. ind. vectors

If $0 \in$ a family of vectors, then this family is lin. dep.

Thus 0 cannot be an element of a basis

Canonical basis for $\mathbb{P}_2$:

$p_1(t) = 1 \qquad p_2(t) = t$

$p_3(t) = t^2 \qquad \Rightarrow \{p_1, p_2, p_3\}$ basis?

(i) ~~We have~~ We can represent the 3 polynomials as

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

Thus $\{p_1, p_2, p_3\}$ is lin. ind.

(ii) Take $p(t) = 5 - 3t + 8t^2$. Then

$p(t) = 5p_1(t) - 3p_2(t) + 8p_3(t)$

$\Rightarrow p = 5p_1 - 3p_2 + 8p_3$

General case

If $p(t) = a + bt + ct^2$,
then

$$p = a\,p_1 + b\,p_2 + c\,p_3$$

$\Rightarrow$ Span $\{p_1, p_2, p_3\}$ is $\mathbb{P}_2$

Def 16

$\Rightarrow$ Dim $(\mathbb{P}_2) = 3$

Example of lin dep. family.

Take

$q_1(t) = 1 - 4t$ $\quad$ $q_2(t) = 1 + t^2$

$q_3(t) = t + 5t^2$ $\quad$ $q_4(t) = 1 + t + t^2$

Thm 15-1

$\Rightarrow$ $\{q_1, q_2, q_3, q_4\}$
lin dep. family

Another basis of $\mathbb{P}_2$

$p_1(x) = 1+x$ $\quad$ $p_2(x) = 2-2x+x^2$

$p_3(x) = 1+x^2$

Question: Is $\{p_1, p_2, p_3\}$ a basis of $\mathbb{P}_2$?

We have seen that $\dim(\mathbb{P}_2)=3$ and we are given 3 elements

$\overset{\text{Thm 17}}{\Longrightarrow}$ we just have to verify that $\{p_1, p_2, p_3\}$ are lin ind.

Compute

$$\det([p_1 \; p_2 \; p_3]) =$$

$$= \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 1 \end{vmatrix} = -3 \neq 0$$

$\Rightarrow \{p_1, p_2, p_3\}$ lin indep

$\overset{\text{Thm 17}}{\Longrightarrow} \{p_1, p_2, p_3\}$ basis of $\mathbb{P}_2$

Example of Null(A) $\subset \mathbb{R}^5$

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & 4 \end{pmatrix}$$

Row-echelon

$$\sim \begin{array}{c} \begin{pmatrix} \boxed{1} & -2 & 0 & -1 & 3 \\ 0 & 0 & \boxed{1} & 2 & -2 \\ 0 & \boxed{0} & 0 & \boxed{0} & \boxed{0} \end{pmatrix} \\ \begin{array}{ccccc} \quad & r & \quad & s & t \end{array} \end{array}$$

Free var: $x_2 = r$, $x_4 = s$, $x_5 = t$

then

$$x_3 = -2s + 2t$$

$$x_1 = 2r + s - 3t$$

$$\text{Null}(A) = \{(2r+s-3t, r, -2s+2t, s, t) : r, s, t \in \mathbb{R}\}$$

Null(A) as a span

Null(A) = $\{(2r+s-3t, r, -2s+2t, s, t); \ldots\}$

$$= \left\{ r\begin{pmatrix}2\\1\\0\\0\\0\end{pmatrix} + s\begin{pmatrix}1\\0\\-2\\1\\0\end{pmatrix} + t\begin{pmatrix}-3\\0\\2\\0\\1\end{pmatrix};\ r,s,t \in \mathbb{R} \right\}$$

We will see that
Null(A) is a 3-dim (Check)
subspace of $\mathbb{R}^5$

Example of Col(A)

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{pmatrix}$$

Then

$$\text{Col}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \\ 10 \end{pmatrix} \right\}$$

↳ subspace of $\mathbb{R}^3$

Rmk It is easy to write Col(A) as a span.

Additional questions:

Basis for Col(A)?

Dim(Col(A)) ? → Prop 20

Example of dim for Col(A), Null(A)

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{pmatrix}$$

Row-echelon

$$\sim \begin{pmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{0} & \boxed{0} \end{pmatrix}$$

2 pivots

2 free variables

Prop20

$\Longrightarrow$ $\mathrm{Dim}(\mathrm{Null}(A)) = 2$

$\mathrm{Dim}(\mathrm{Col}(A)) = 2$

Rmk In order to get some information about Row(A), we could simply write

$$\mathrm{Row}(A) = \mathrm{Col}(A^T)$$

However, there is a simpler solution → Thm 21

Example of Row(A)

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{pmatrix}$$

$$\sim \begin{pmatrix} \boxed{1 \quad 2 \quad 3 \quad 4} \\ \boxed{0 \quad 1 \quad 2 \quad 3} \\ \cancel{0 \quad 0 \quad 0 \quad 0} \end{pmatrix}$$

Basis for Row(A)

Rmk: $\mathrm{Dim}(\mathrm{Row}(A)) = 2$

Example of Col(A)

$$A = \begin{pmatrix} \boxed{1} & \boxed{2} & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{pmatrix}$$

$$\sim \begin{pmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thm 22: Basis for Col(A) is

$$\left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \right\}$$