Intro to 2nd order diff eq

Review of Col, Row, Null

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{pmatrix}$$

$$\sim \begin{pmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{0} & \boxed{0} \end{pmatrix}$$

Rank(A)
=
Dim (Col(A)) = 2
= Dim (Row(A))

Dim (Null(A))
= 2

2 elements

Basis for Col(A) : $\left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \right\}$

Basis Row(A) : $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\}$ 2 elements

verifying the rank thm

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{pmatrix} \to \underset{n}{\underline{\underline{4}}} \text{ columns}$$

Here we have seen

$\text{Rank}(A) = 2$

$\text{Dim}(\text{Null}(A)) = 2$

Thus

$$\text{Rank}(A) + \text{Dim}(\text{Null}(A)) = n$$
$$2 \quad + \quad 2 \quad = \quad 4$$

Solutions for 1st order diff

$$y' = ay$$

General solution: $y = ce^{\alpha t}$

Basic idea: for 2nd order) $ay''+by'+cy=0$
diff eq which are linear with constant coeff, try to find solution of the same form i.e

$$y = e^{\alpha t}$$

Initial conditions

1st order diff => integrate once
=> producing 1 constant
=> need 1 initial condition

2nd order diff => integrate twice
=> producing 2 constants
=> need 2 initial conditions

Vector space $\mathbb{R}^2$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$

Generic element of $\mathbb{R}^2$: $\begin{pmatrix} x \\ y \end{pmatrix}$

Basis for $\mathbb{R}^2$

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

2 elements

$\text{Dim}(\mathbb{R}^2) = 2$

Thm

(i) Any basis in $\mathbb{R}^2$ has 2 elements

(ii) If $v_1, v_2 \in \mathbb{R}^2$ are lin. indep. then $\{v_1, v_2\}$ is a basis

Solution set S of $ay'' + by' + cy = 0$

S is a vector space

$\dim(S) = 2$

Any basis of S has 2 elements

If y_1, y_2 solve the equation
and y_1, y_2 lin indep

$\Rightarrow \{y_1, y_2\}$ basis of S

$\Rightarrow$ General solution:

$$y = c_1 y_1 + c_2 y_2$$

Question: given y_1, y_2 solutions of $ay'' + by' + cy = 0$, how do I know that y_1, y_2 are lin indep?

Definition of Wronskian

y_1, y_2 are lin indep

$\iff$ $y_1 y_2' - y_1' y_2 \neq 0$ at least for one t

$\iff$ $\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \equiv W[y_1, y_2](t)$

Equation $y'' + y' - 6y = 0$

Claim : $y_1 = e^{2x}$ solves the eq

$y_1 = e^{2x}$

$y_1' = 2e^{2x}$

$y_1'' = 4e^{2x}$

Thus

$y_1'' + y_1' - 6y_1$

$= 4e^{2x} + 2e^{2x} - 6e^{2x}$

$= 0 \Rightarrow y_1$ solves the eq.

In the same way

$y_2 = e^{-3x}$ solves the eq.

Q: Are $y_1 = e^{2x}$, $y_2 = e^{-3x}$ lin indep?

We compute

$W[y_1, y_2](x)$

$$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix}$$

$$= -3e^{-x} - 2e^{-x}$$

$$= -5e^{-x} \neq 0 \quad \text{for all } x$$

$\Rightarrow$ y_1, y_2 are lin indep

$\Rightarrow$ Gen sol: $y = c_1 e^{2x} + c_2 e^{-3x}$

Example of equation

$$y'' + 5y' + 6y = 0$$

Aux. eq.

$$r^2 + 5r + 6 = 0$$

Quadratic formula: for

$$ar^2 + br + c = 0$$

$$\Delta = b^2 - 4ac,$$

then $$r_1 = \frac{-b + \sqrt{\Delta}}{2a}$$

$$r_2 = \frac{-b - \sqrt{\Delta}}{2a}$$

Application r^2+5r+6

$\Delta = 5^2 - 4\times1\times6 = 1$

Thus

$r_1 = \dfrac{-5+1}{2} = -2$

$r_2 = \dfrac{-5-1}{2} = -3$

Fundamental solutions:

$y_1 = e^{-2t}$ $\quad$ $y_2 = e^{-3t}$

General solution

$y = c_1 e^{-2t} + c_2 e^{-3t}$

Initial cond: $y(0) = 2 \qquad y'(0) = 3$

$$y = c_1 e^{-2t} + c_2 e^{-3t}$$

$$y' = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

We get, at $t=0$

$$\begin{cases} c_1 + c_2 = 2 \\ -2c_1 - 3c_2 = 3 \end{cases}$$

$$\Leftrightarrow \overbrace{\begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix}}^{\det(A) = -1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Thus with Cramer's rule,

$$c_1 = \frac{1}{-1} \begin{vmatrix} 2 & 1 \\ 3 & -3 \end{vmatrix} = 9$$

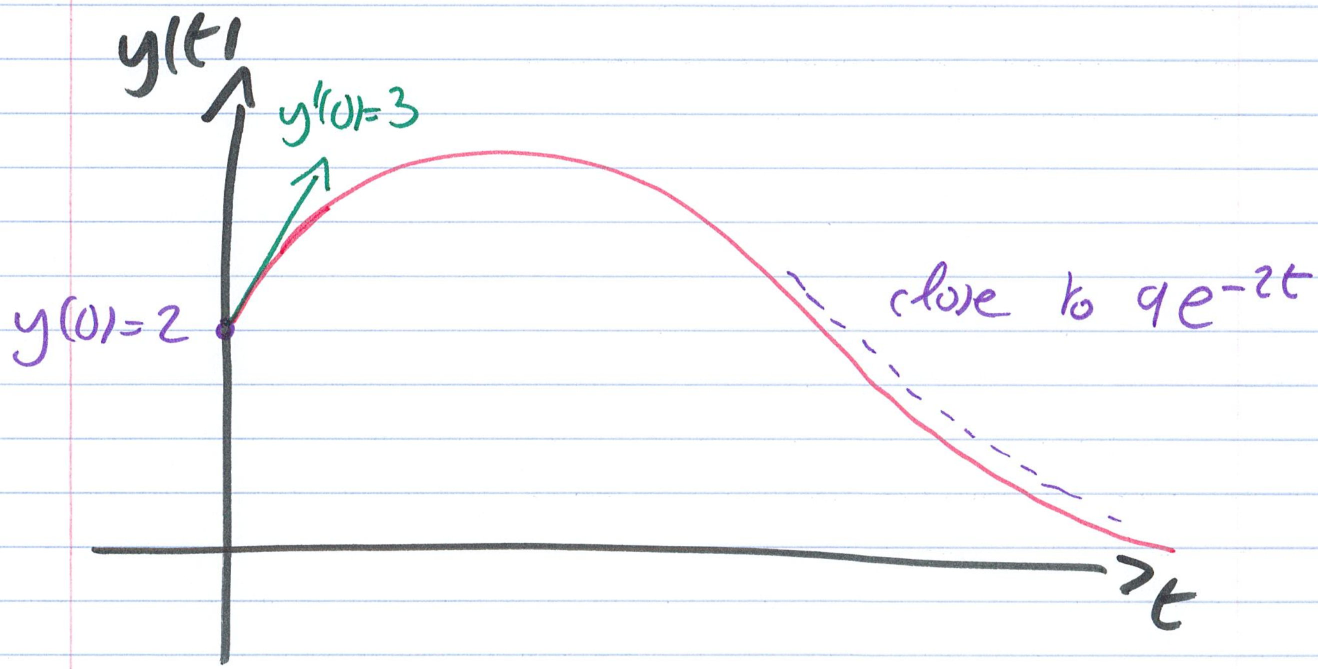
$$c_2 = - \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} = -7$$

Unique solution

$$y = 9\,e^{-2t} - 7\,e^{-3t}$$

Dominant term as $t \to \infty$

Both are decaying exponentially



Another eq

$$y'' + y' + \frac{37}{4} y = 0$$

Aux eq

$$r^2 + r + \frac{37}{4} = 0$$

$$\Delta = 1^2 - 4 \times 1 \times \frac{37}{4} = -36$$

Thus

$$r_1 = \frac{-1 + 6i}{2} = -\frac{1}{2} + 3i$$

$$r_2 = \frac{1}{2} - 3i$$

Gen sol

$$y = c_1 e^{-\frac{t}{2}} \cos(3t)$$

$$+ c_2 e^{-\frac{t}{2}} \sin(3t)$$