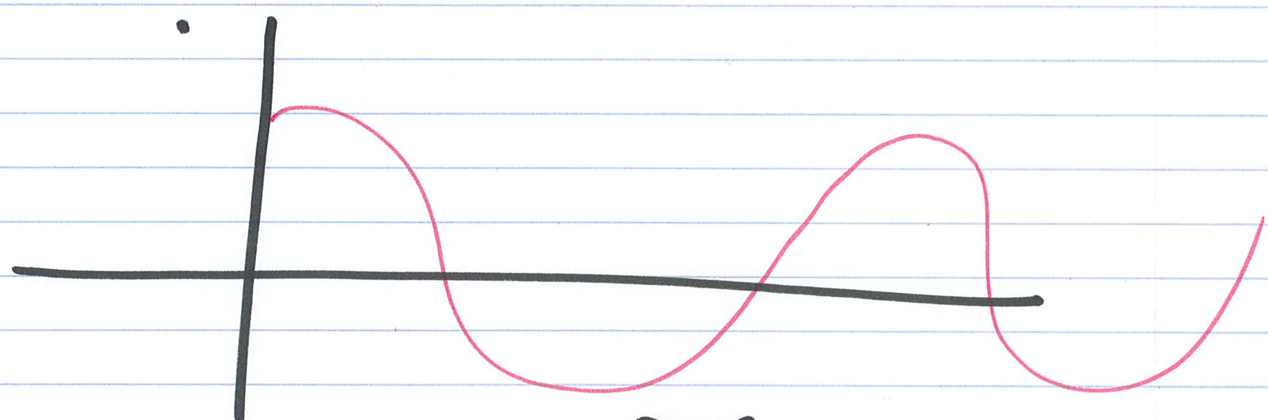
Undetermined coeff

Mechanical vibrations

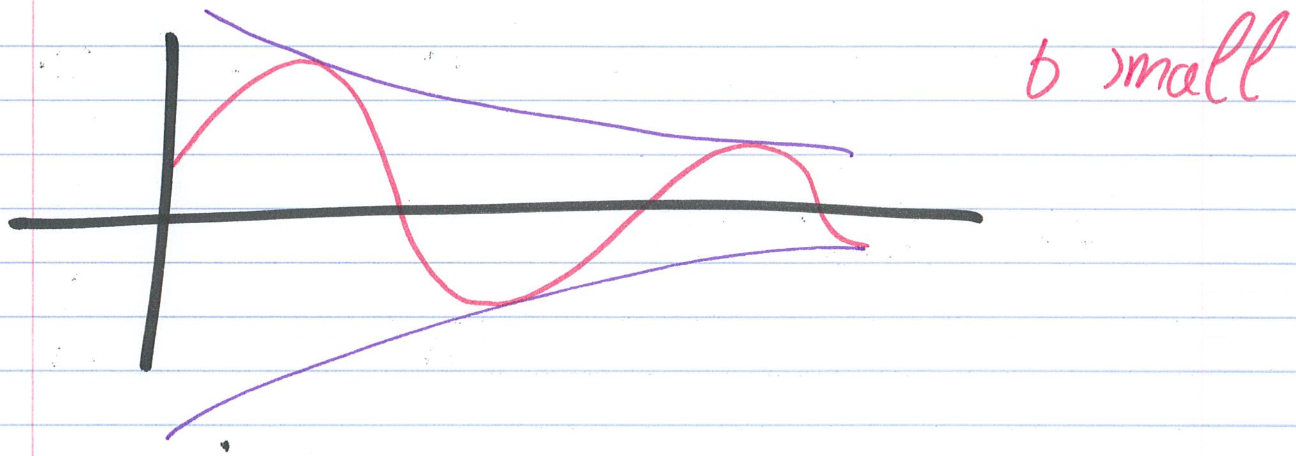
$$m y'' + b y' + k y = 0$$

If $b = 0$, solution of the form

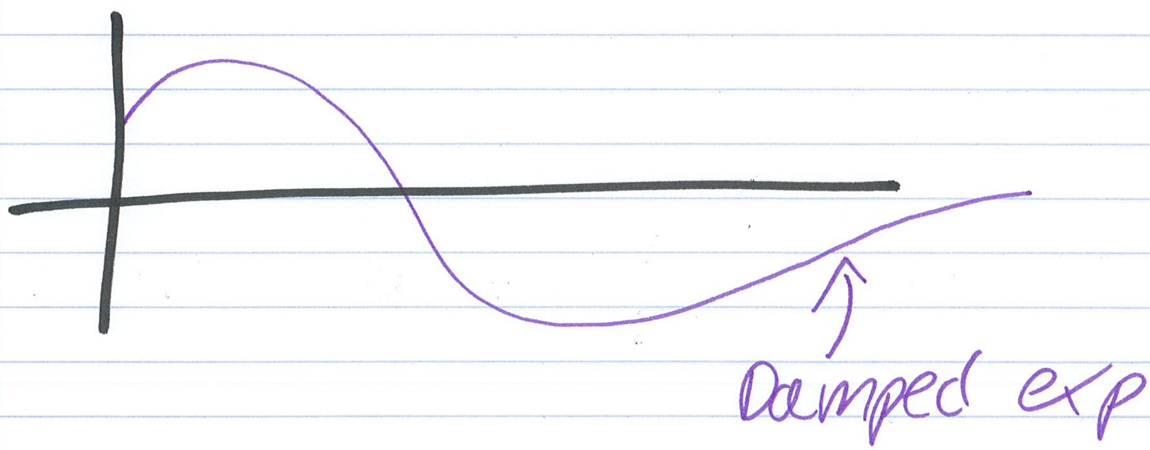


with $\omega = \sqrt{\frac{k}{m}}$

If $b > 0$, solution can either be



or



End of program for Midterm 2

Higher order eq

$$\underbrace{a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y}_{Ly} = F(x)$$

Main question after Thm 10

How to compute y_p ?

Today: we will see how to compute y_p for specific classes of functions F

Equation

~~Function~~ with facing term

$$\underbrace{y'' - 3y' - 4y}_{Ly} = \underbrace{2\sin(t)}_{f(t)}$$

1 Hom eq

$$y'' - 3y' - 4y = 0$$

$$P(r) = r^2 - 3r - 4$$

Trivial root $r_1 = -1$

Then $r_2 = 4$

(product $r_1 r_2 = -4$)

Then fund solutions are

$$y_1 = e^{-t}$$

$$y_2 = e^{4t}$$

2 Particular solution. we have

$$f(t) = 2\sin(t)$$

$\Rightarrow$ we will have $y_p = a\sin(\cancel{2}t) + b\cos(\cancel{2}t)$

$$y_p = a\sin(\cancel{2}t) + b\cos(\cancel{2}t) \quad \times(-4)$$

$$y_p' = -\cancel{2}b\sin(\cancel{2}t) + \cancel{2}a\cos(\cancel{2}t) \quad \times(-3)$$

$$y_p'' = -a\sin(t) - b\cos(t) \quad \times 1$$

$$(-a + 3b - 4a)\sin(t)$$

$$+ (-b - 3a - 4b)\cos(t)$$

$$\Rightarrow \quad y_p'' - 3y_p' - 4y_p$$

$$= (-5a + 3b)\sin(t)$$

$$+ (-3a - 5b)\cos(t)$$

$(-5a+3b)\sin(t)+(-3a-5b)\cos(t)$

<u>we wish</u> $y_p''-3y_p'-4y_p$ to be $2\sin(t)$. we get a system

$$-5a+3b=2$$
$$-3a-5b=0$$

$\det = 34$

$$\Leftrightarrow \begin{pmatrix}-5 & 3\\ -3 & -5\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}2\\ 0\end{pmatrix}$$

$$a=\frac{1}{34}\begin{vmatrix}2 & 3\\ 0 & -5\end{vmatrix}=-\frac{5}{17}$$

$$b=\frac{1}{34}\begin{vmatrix}-5 & 2\\ -3 & 0\end{vmatrix}=\frac{3}{17}$$

<u>Thus</u>

$$y_p=-\frac{5}{17}\sin(t)+\frac{3}{17}\cos(t)$$

General solution

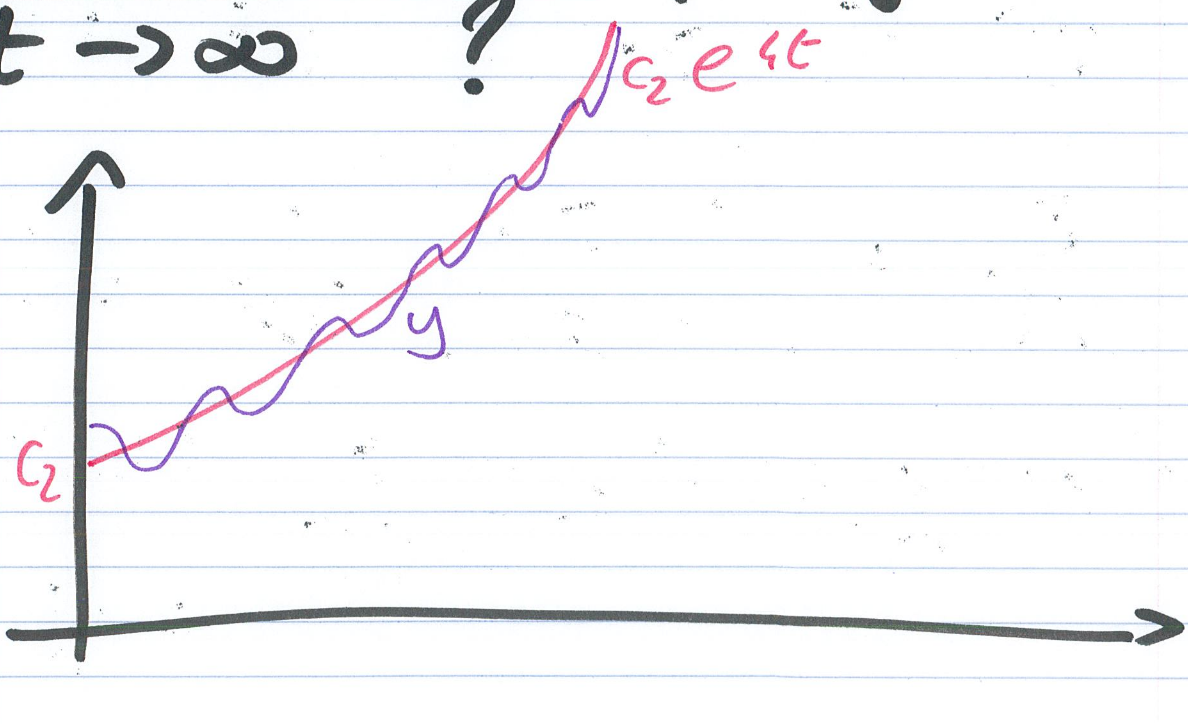
$$y = c_1 \underbrace{e^{-t}}_{\to 0 \text{ as } t \to \infty} + c_2 \underbrace{e^{4t}}_{\to +\infty \text{ as } t\to\infty}$$

$$\underbrace{+ \frac{3}{17}\cos(t) - \frac{5}{17}\sin(t)}_{\to \text{ bounded}}$$

Then c_1, c_2 can be computed if we are given e.g

$$y(0) = \alpha \quad , \; y'(0) = \beta$$

Dominant term for y as $t \to \infty$?



Example

$$y'' - 3y' - 4y = -8e^{t}\cos(2t)$$

1. Hom eq

$y_1 = e^{-t}$ $y_2 = e^{4t}$

2. Particular solution.

we will assume

$$y_p = a\cos(2t)e^{t} + b\sin(2t)e^{t}$$

Computation for y_p

$\times(-4)$

$$y_p = a\cos(2t)e^t + b\sin(2t)e^t$$

$\times(-3)$

$$y'_p = (a+2b)\cos(2t)e^t + (-2a+b)\sin(2t)e^t$$

$$y''_p = \left(\begin{array}{l} a+2b \\ -2a+2b \end{array}\right)\cos(2t)e^t + \left(\begin{array}{l} -2a-4b \\ -2a+b \end{array}\right)\sin(2t)e^t$$

$\times(1)$

$$y''_p = (-3a+4b)\cos(2t)e^t + (-4\underset{a}{b}-3b)\sin(2t)e^t$$

$$(-4a-3a-6b-3a+4b)\cos(2t)e^t$$

$$+(-4b+6a-3b-4a-3b)\sin(2t)e^t$$

Thus

$$y''_p - 3y'_p + 4y_p$$

$$= (-10a-2b)\cos(2t)e^t$$

$$+ (2a-10b)\sin(2t)e^t$$

↳ we wish this to be $-8e^t\cos(2t)$

System for a, b:

$$\begin{cases} -10a - 2b = -8 \\ 2a - 10b = 0 \end{cases}$$

We find (check)

$$a = \frac{10}{13}, \quad b = \frac{2}{13}$$

$$\Rightarrow y_p = \left(\frac{10}{13}\cos(2t) + \frac{2}{13}\sin(2t)\right)e^t$$

General solution

$$y = c_1 e^{-t} + c_2 e^{4t}$$

still dominant as $t \to \infty$

$$+ \left(\frac{10}{13}\cos(2t) + \frac{2}{13}\sin(2t)\right)e^t$$

Example with elaboration

$$y^{(3)} - 3y^{(2)} + 3y' - y = 4e^t$$

1 Hom. eq

$$y^{(3)} - 3y^{(2)} + 3y' - y = 0$$

$$P(r) = r^3 - 3r^2 + 3r - 1$$

$$= (r-1)^3 \Rightarrow 1 \text{ root with}$$

$$s = 3$$

multiplicity

$$y_1 = e^t, \quad y_2 = te^t, \quad y_3 = t^2 e^t$$

2 Guess for y_p

$$y_p = t^3 \, a e^{1t}$$

Then

$y_p = a t^3 e^t$ $\times(-1)$

$y_p' = a t^3 e^t + 3a t^2 e^t$ $\times(3)$

$y_p'' = a t^3 e^t + 6a t^2 e^t + 6a t e^t$ $\times(-3)$

$y_p^{(3)} = a t^3 e^t + 9a t^2 e^t + 18a t e^t + 6a e^t$ $(\times 1)$

We get

$y_p^{(3)} - 3y_p'' + 3y_p' - y_p = 6a e^t$

We wish this to $4e^t$. We get

$$6a = 4 \Rightarrow a = \frac{2}{3}$$

General solution

$$y_p = \left(c_1 + c_2 t + c_3 t^2 + \frac{2}{3} t^3\right) e^t$$

Dominant term