Variation of parameters

3rd order diff eq

$$y''' - 4y' = t + 3\cos(t) + e^{-2t}$$

1 Hom eq : $y''' - 4y' = 0$

$$P(r) = r^3 - 4r$$
$$= r(r^2 - 4)$$
$$= r(r-2)(r+2)$$

Roots : $0, 2, -2$

Fund sol : $1, \quad e^{2t}, \quad e^{-2t}$

$y''' - 4y' = t + 3\cos(t) + e^{-2t}$ (2)

Roots: 0, 2, -2

(2) For y_p, we can split the rhs into 3 pieces

$e^{0t} \times t$, $3\cos(t)$, e^{-2t}

Then we just add up all the guesses for the 3 pieces. We get a guess of the form

$$y_p = t(a_0 + a_1 t)$$

$$+ \quad b\cos(t) + c\sin(t)$$

$$+ \quad d t e^{-2t}$$

Then compute $y_p''' - 4y_p'$ in order to identify a_0, a_1, b, c, d

General idea for var of cst

(i) Start from y_1, y_2 fund solutions of the hom. eq (2nd order eq)

(ii) Try to get a solution y_p starting from y_1, y_2. In fact ~~ix~~ we will write

$$y_p = y_1 u_1 + y_2 u_2 ,$$

where u_1, u_2 are functions instead of constants

↳ variation of the constant

$\sec(x) = \frac{1}{\cos(x)}$

Eq : $1 \times y'' + y = \underbrace{\sec(x)}_{f}$

(1) Hom eq $\quad y'' + y = 0$

Fund sol: $\quad p(r) = r^2 + 1$

Roots: $r_1 = i \quad r_2 = -i$

Thus $\quad y_1 = \cos(x) \quad y_2 = \sin(x)$

(2) Find y_p under the form

$y_p = y_1 u_1 + y_2 u_2$

Then u_1', u_2' solve

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = \sec(x) \end{cases}$$

Write the system

$$\begin{cases} \cos(x)\, u_1' + \sin(x)\, u_2' = 0 \\ -\sin(x)\, u_1' + \cos(x)\, u_2' = \sec(x) \end{cases}$$

$$\Leftrightarrow \underbrace{\begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix}}_{\det = \cos^2(x) + \sin^2(x) = 1} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \sec(x) \end{pmatrix}$$

Thus according to Cramer, we have

$$u_1' = \frac{1}{1} \begin{vmatrix} 0 & \sin(x) \\ \cancel{\sin}\ \sec(x) & \cos(x) \end{vmatrix}$$

$$= -\frac{\sin(x)}{\cos(x)}$$

$$u_2' = \frac{1}{1} \begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & \sec(x) \end{vmatrix} = 1$$

$u_1' = \frac{-\sin(x)}{\cos(x)}$ $\quad u_2' = 1$

$\frac{v'}{v}$ with $v = \cos(x)$

Compute u_1, u_2

$$u_1 = \int u_1' \, dx = + \int \frac{-\sin(x)}{\cos(x)} \, dx$$

$$= \ln(|\cos(x)|) \quad (+C)$$

$$u_2 = \int u_2' \, dx = \int dx$$

$$= x \quad (+C)$$

Write y_p

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \ln(|\cos(x)|)\cos(x) + x\sin(x)$$

General solution

$$y = c_1 y_1 + c_2 y_2 + y_p$$

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(|\cos(x)|) \cos(x) + x \sin(x)$$

Rmk This method is due to Green

17th
18th
19th

Eq

$$y'' + 4y' + 4y = \overbrace{e^{-2x} \ln(x)}^{f \text{ not in our table of guess}}$$

1 Hom eq : $y'' + 4y' + 4y = 0$

$$P(r) = r^2 + 4r + 4$$

$$= (r+2)^2$$

Fund sol : $y_1 = e^{-2x}$

$y_2 = x e^{-2x}$

2 We look for $y_p = y_1 u_1 + y_2 u_2$ with u_1', u_2' solving

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = e^{2x} \ln(x) \end{cases}$$

Write the system

$$\begin{cases} \overbrace{e^{-2x}}^{y_1} u_1' + \overbrace{x e^{2x}}^{y_2} u_2' = 0 \\ \overbrace{-2 e^{-2x}}^{y_1'} u_1' + \overbrace{(-2x+1)e^{-2x}}^{y_2'} u_2' = e^{-2x} \ln(x) \end{cases}$$

$$\iff \begin{cases} u_1' + x\, u_2' = 0 \\ -2u_1' + (-2x+1)\, u_2' = \ln(x) \end{cases}$$

$$\iff \underbrace{\begin{pmatrix} 1 & x \\ -2 & -2x+1 \end{pmatrix}}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \ln(x) \end{pmatrix}$$

$\det = -2x + 1 + 2x = 1$

$(\ln(x))' = \frac{1}{x}$

Cramer's rule :

$$u_1' = \frac{1}{1}\begin{vmatrix} 0 & x \\ \ln(x) & -2x+1 \end{vmatrix}$$

$$u_1' = -x\ln(x)$$

$$u_2' = \frac{1}{1}\begin{vmatrix} 1 & 0 \\ -2 & \ln(x) \end{vmatrix}$$

$$u_2' = \ln(x)$$

Compute U_2

$$U_2 = \int u_2'\,dx$$

$$= +\int \underbrace{1}_{u'} \times \overbrace{\ln(x)}^{v}\,dx$$

$$\overset{\text{ibp}}{=} uv - \int u v'$$

$$= x\ln(x) - \int x \times \frac{1}{x}\,dx$$

$= x\ln(x) - x\ (+c)$

Compute u_1

$$u_1 = \int u_1' \, dx$$

$$= -\int \overset{u'}{x} \, \overset{v}{\ln|x|} \, dx$$

$$= -\left(\frac{1}{2} x^2 \ln|x| \right.$$

$$\left. + \int \frac{1}{2} x^2 \times \frac{1}{x} \, dx \right)$$

$$= -\frac{1}{2} x^2 \ln|x| + \frac{1}{2} \int x \, dx$$

$$= -\frac{1}{2} x^2 \ln|x| + \frac{1}{4} x^2 \; (+C)$$

Write y_p

$$y_p = y_1 u_1 + y_2 u_2$$

$$= e^{-2x} \left(-\frac{1}{2} x^2 \ln|x| + \frac{1}{4} x^2 \right.$$

$$\left. + x (x \ln x - x) \right)$$

General solution

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$
$$+ e^{-2x}\left(-\frac{1}{2}x^2 \ln(x) + \frac{1}{4}x^2 \right.$$
$$\left. + x\,(x \ln(x) - x) \right)$$

$$= e^{-2x} \left\{ c_1 + c_2 x \right.$$
$$\left. + \frac{1}{2} x^2 \ln(x) - \frac{3}{4} x^2 \right\}$$