First order systems

Eigenvalues. Let A a $n \times n$ matrix. Then v is an eigenvector with eigenvalue λ if

$$Av = \lambda v$$

In order to get the eigenvalues, we compute

$$\det(A - \lambda I)$$ polynomial in λ

Then eigenvalues are roots of this polynomial

Once eigenvalues are obtained, the eigenvectors are computed by solving $(A - \lambda I)v = 0$ lin. hom. system

Rmk on spring-mass system
we get a sys coupled
system describing x_1, x_2

eq for x_1 involves x_2
" " x_2 " x_1

2nd observation Any linear
higher n-th order eq can
be written as a system
of first order diff eq. in $\mathbb{R}^n$

2nd-order example

$$y'' + \frac{1}{8} y' + y = 0 \qquad y = y(t)$$

Set $x_1(t) = y$

$x_2(t) = y' \Rightarrow x_2' = y''$

Then

$(y)' = x_1'(t) = x_2(t) = y'$

$x_2'(t) + \frac{1}{8} x_2(t) + x_1(t) = 0$

Thus

linear syst of 1st order eq. in $\mathbb{R}^2$

$$\left.\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - \frac{1}{8} x_2 \end{aligned}\right\}$$

From system in $\mathbb{R}^2$ to 2^{nd} order diff eq

$$\begin{cases} x' = -2y \\ y' = \frac{1}{2}x \end{cases}$$

Then differentiate the 1^{st} eq:

$$x'' = -2y'$$

$$\overset{2^{nd}\text{ eq}}{=} -2\left(\frac{1}{2}x\right)$$

$$= -x$$

The eq for x is thus

$$x'' + x = 0$$

Thus

Eq: $x'' + x = 0$ $P(r) = r^2 + 1$
roots: $\pm i$

Gen solution:

$$x = c_1 \cos(t) + c_2 \sin(t)$$

or

$$x = A \cos(t - \varphi)$$

In order to solve for y,
we write (1st eq)

$$y = -\frac{1}{2} x' = \frac{1}{2} A \sin(t - \varphi)$$

Conclusion. It is equivalent to solve

- 1st order system in $\mathbb{R}^2$
- 2nd order eq. in $\mathbb{R}$

Graph for general solution. We have

$$\begin{cases} x = A\cos(t-\varphi) \\ y = \frac{1}{2}A\sin(t-\varphi) \end{cases}$$

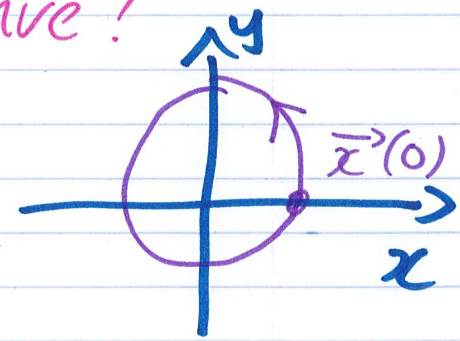
↳ curve in $\mathbb{R}^2$

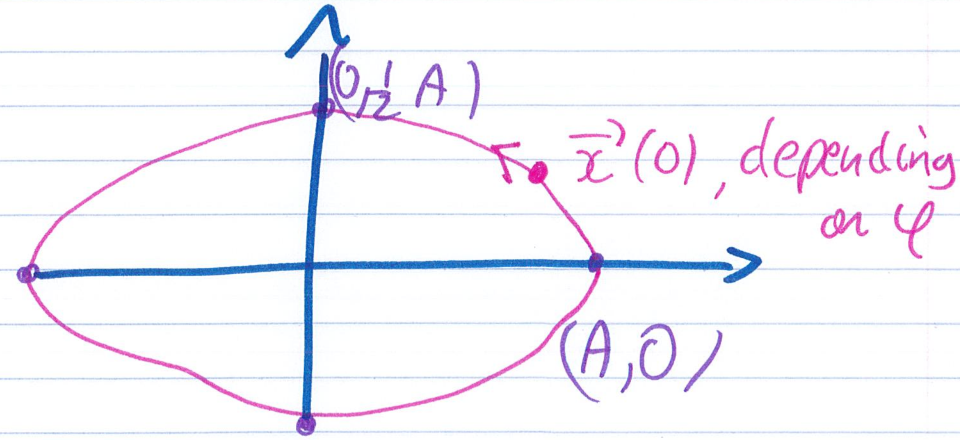
Question: If we have

$$\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$$

→ what is the curve?

↳ $x^2+y^2=1$





Ex of 1^{st} order system

$$\begin{cases} x' = y \\ y' = 2x + y \end{cases}$$

Differentiate 1^{st} eq.

$$x'' = y' \overset{2^{nd} eq}{=} 2x + \underbrace{y}_{= x' \ (1^{st} eq)} = 2x + x'$$

We get a 2^{nd} order diff eq

$$x'' - x' - 2x = 0$$

$$P(r) = r^2 - r - 2$$

Gen sol: $x = c_1 e^{-t} + c_2 e^{2t}$

Then $y = x' = -c_1 e^{-t} + 2c_2 e^{2t}$

$\hookrightarrow$ 1^{st} eq

$x = c_1 e^{-t} + c_2 e^{2t}$
$y = -c_1 e^{-t} + 2c_2 e^{2t}$

We get, in $\mathbb{R}^2$

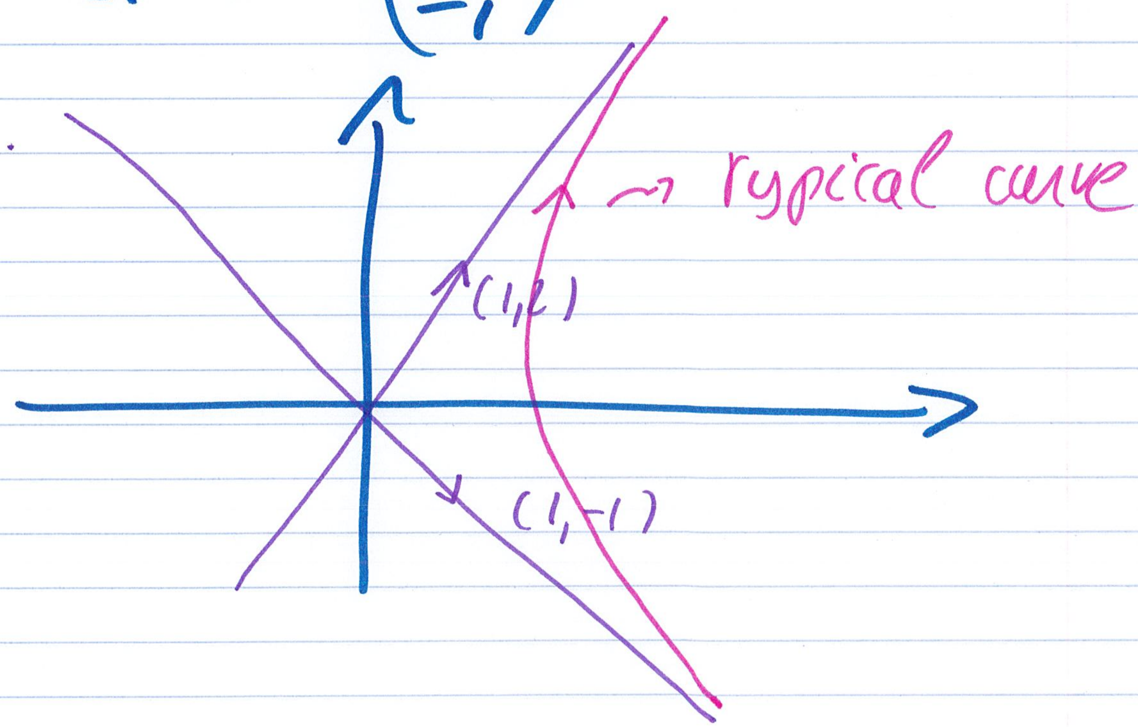
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Dominant term as $t \to +\infty$

$$c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$ → dominant direction

Dominant term as $t \to -\infty$

$$c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ → dominant direction



System as a matrix system

$$\begin{cases} x' = & y \\ y' = 2x & + y \end{cases}$$

can be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Example of initial cond.

$$\begin{cases} x_1' = x_1 + 2x_2 \\ x_2' = 2x_1 - 2x_2 \end{cases}$$

we will see that a general sol. to this system is

$$\begin{cases} x_1 = c_1 e^{-3t} + c_2 e^{2t} \\ x_2 = -2c_1 e^{-3t} + \frac{1}{2} c_2 e^{2t} \end{cases}$$

In order to get a unique sol, we are given $x_1(0) = 1$ $x_2(0) = 0$ then we get

$$\begin{cases} 1 = c_1 + c_2 \\ 0 = -2c_1 + \frac{1}{2} c_2 \end{cases}$$

system for c_1, c_2

Then unique sol is

$$x_1 = \frac{1}{5} e^{-3t} + \frac{4}{5} e^{2t}$$

$$x_2 = -\frac{2}{5} e^{-3t} + \frac{2}{5} e^{2t}$$

Example of Wronskian in $\mathbb{R}^2$

consider

$$\vec{x}_1(t) = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix} \qquad x_2(t) = \begin{pmatrix} 3\sin(t) \\ \cos(t) \end{pmatrix}$$

we wish to know if those functions are linearly dependent. we compute

$$W[x_1, x_2](t) = \begin{vmatrix} e^t & 3\sin(t) \\ 2e^t & \cos(t) \end{vmatrix}$$

$$= e^t \cos(t) - 2e^t \times 3\sin(t)$$

$$= e^t (\cos(t) - 6\sin(t))$$

At $t_0 = 0$, we have

$$W[x_1, x_2](0) = e^0 (1 - 0) = 1 \neq 0$$

$\Rightarrow$ According to Thm 2, $\{x_1, x_2\}$ are lin. indep.