Eigenvalue method

Main remaining question for systems: how to compute the fund. sol.? ~~when~~

Today: if coefficients are constant, we can use the eigenvalue decomp. of A for a system

$$\vec{x}\,' = A\vec{x}$$

$a^2 - b^2 = (a-b)(a+b)$

Ex of system in $\mathbb{R}^2$

$x' = Ax$ with $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

Then

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix}$$

$$= (\lambda - 1)^2 - 4 = (\lambda-1)^2 - 2^2$$

$$= (\lambda - 3)(\lambda + 1)$$

Thus $\lambda_1 = 3$, $\lambda_2 = -1$

$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

Eigenvector for $\lambda_1 = 3$

$$(A - 3I)u = 0$$

$$\Leftrightarrow \begin{pmatrix} -2 & 1 \\ \cancel{4} & \cancel{-2} \end{pmatrix} u = 0$$

$$\Leftrightarrow -2u_1 + u_2 = 0 \Leftrightarrow u_2 = 2u_1$$

Thus take

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Eigenvector for $\lambda_2 = -1$

$$(A + I)u = 0 \Leftrightarrow \begin{pmatrix} 2 & 1 \\ \cancel{4} & \cancel{2} \end{pmatrix} u = 0$$

$$\Leftrightarrow u_2 = -2u_1$$

Take $\vec{u}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Summary: we have found

$\lambda_1 = 3 \qquad \vec{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda_2 = -1 \qquad \vec{u}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Apply Thm 6 we get

$$\vec{x}_1 = e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\vec{x}_2 = e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

General solution

$$\vec{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{3t} + c_2 e^{-t} \\ 2c_1 e^{3t} - 2c_2 e^{-t} \end{pmatrix}$$

Dominant term $c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ as $t \to +\infty$

Additional question. we have

$\vec{x}_1 = e^{3t}\begin{pmatrix}1\\2\end{pmatrix}$ $\vec{x}_2 = e^{-t}\begin{pmatrix}1\\-2\end{pmatrix}$

~~Can~~ According to Thm 6, $\vec{x}_1$ and $\vec{x}_2$ are lin. indep. (fund solutions). Can we check that directly.

$$W[\vec{x}_1, \vec{x}_2](t)$$

$$= \begin{vmatrix} e^{3t} \times 1 & e^{-t} \\ e^{3t} \times 2 & -2e^{-t} \end{vmatrix}$$

$$= e^{2t}(-2 - 2)$$

$$= -4e^{2t} \neq 0$$

$\Rightarrow \vec{x}_1, \vec{x}_2$ are lin. indep.

Tank system

Eq for x_1 = quant. of salt in tank 1

$$x_1' = \text{rate in} - \text{rate out}$$

fresh water in

concentration in tank 1

$$= r \times 0 - r \times \frac{x_1}{V_1}$$

$\equiv k_1$

$$x_1' = -\frac{r}{V_1} x_1$$

Eq for x_2 = qty salt in tank 2

$$x_2' = k_1 x_1 - k_2 x_2$$

$= r/V_2$

Eq for x_3

$$x_3' = k_2 x_2 - k_3 x_3$$

System. Set $\vec{x} = (x_1, x_2, x_3)$
Then

$$\vec{x}' = \underbrace{\begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & -k_3 \end{pmatrix}}_{A} \vec{x}$$

Example

$r = 10$, $V_1 = 20$, $V_2 = 40$, $V_3 = 50$

Then $k_1 = \frac{r}{V_1} = \frac{1}{2}$

$k_2 = \frac{r}{V_2} = \frac{1}{4}$ $\qquad k_3 = \frac{1}{5}$

$$A = \begin{pmatrix} -1/2 & 0 & 0 \\ 1/2 & -1/4 & 0 \\ 0 & 1/4 & -1/5 \end{pmatrix}$$

Eigenvalue dec

$\lambda_1 = -1/2 \qquad \vec{u}_1 = \begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix}$

$\lambda_2 = -1/4 \qquad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$

$\lambda_3 = -\frac{1}{5} \qquad \vec{u}_3 \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Thus general sol is

$$\vec{x} = c_1 e^{-t/2}\begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix} + c_2 e^{-\frac{1}{4}t}\begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} + c_3 e^{-t/5}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lim_{t \to \infty} \vec{x}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

not a surprise (fresh water added)

We have seen

$$\vec{x}(t) = c_1 e^{-t/2} \begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix} + c_2 e^{-\frac{1}{2}t} \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$$

$$+ c_3 e^{-t/5} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Initial data

$$\vec{x}(0) = \begin{pmatrix} 15 \\ 0 \\ 0 \end{pmatrix}$$

System for c_1, c_2, c_3

$$\begin{pmatrix} 3 & 0 & 0 \\ -6 & 1 & 0 \\ 5 & -5 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 15 \\ 0 \\ 0 \end{pmatrix}$$

we get (row-echelon)

$c_1 = 5, \quad c_2 = 30, \quad c_3 = 125$

Unique solution

$$\vec{x}(t) = 5\, e^{-t/2} \begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix} + 30\, e^{-\frac{1}{4}t} \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} + 125\, e^{-\frac{1}{5}t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Ex: quantity of salt in Tank 2, time t

$$x_2(t) = -30\, e^{t/2} + 30\, e^{-\frac{1}{4}t}$$

System with $\lambda \in \mathbb{C}$

$$x' = \underbrace{\begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}}_{A} x$$

$$\det(A - \lambda I) = \begin{vmatrix} -\frac{1}{2}-\lambda & 1 \\ -1 & -\frac{1}{2}-\lambda \end{vmatrix}$$

$$= \left(\lambda + \frac{1}{2}\right)^2 + 1$$

Roots : $\lambda_1 = -\frac{1}{2} + i$

$\left(\lambda_2 = -\frac{1}{2} - i\right)$

Eigen vector for λ_1

$$\left(A - \left(-\frac{1}{2} + i\right) I\right) u = 0$$

$$\Leftrightarrow \begin{pmatrix} -i & 1 \\ \cancel{-1} & \cancel{-i} \end{pmatrix} u = 0 \Leftrightarrow u_2 = i u_1$$

Take $\vec{u}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

Summary. We have found

$$\lambda_1 = \underbrace{-\tfrac{1}{2}}_{\alpha} + i \times \underbrace{1}_{\beta} \qquad \vec{u}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{a}} + i \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{b}}$$

We get (Thm 7)

$$\vec{x}_1 = e^{-\frac{1}{2}t}\left(\cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cancel{+} - \sin(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$\vec{x}_2 = e^{-\frac{1}{2}t}\left(\sin(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

Gen solution

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

$$= e^{-\frac{1}{2}t}\begin{pmatrix} c_1 \cos(t) + c_2 \sin(t) \\ -c_1 \sin(t) + c_2 \cos(t) \end{pmatrix}$$

Eq for x_2:

$$x_2(t) = e^{-\frac{1}{2}t}\left(c_1 \sin(t) + c_2 \cos(t)\right)$$