Multiple eigenvalue sol.

System with $\mathbb{C}$ eigenvalues

$$x' = Ax \qquad A = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}$$

Eigenvalue decomposition for A

$$\lambda_1 = \overbrace{-\frac{1}{2}}^{\alpha} + i \times \overbrace{1}^{\beta}$$

and $\lambda_2 = \bar{\lambda}_1$, $\vec{u}_2 = \vec{\bar{u}}_1$

$$\vec{u}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{a}} + i \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{b}}$$

According to Thm 7, we get

$$\vec{x}_1 = e^{-\frac{1}{2}t}\left(\cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$\vec{x}_2 = e^{-\frac{1}{2}t}\left(\sin(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

Then general solution is

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Rmk For systems of the form $x' = Ax$ we have seen two cases

(i) A has real eigenvalues, all distinct

(ii) A has complex eigenvalues

Today: what happens ~~wet~~ when we have repeated eigenvalues?

Example of matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

Then $\det(A - r I)$

$$= \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix}$$

$$= (r-1)(r-3) + 1$$

$$= r^2 - 4r + 4 = (r-2)^2$$

Thus $r=2$ is a double eigenvalue

Rmk We know that there will be 1 eigenvector only. If we had 2 eigenvectors for $r=2$, then we would have $A = 2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Eigenvector for $r=2$

$$A - 2I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$(A-2I)\vec{v}_1 = 0$$

$$\Leftrightarrow \quad v_1 + v_2 = 0 \quad \Leftrightarrow \quad v_2 = -v_1$$

we take $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

One fundamental sol is given by

$$\vec{x}_1(t) = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

How do we get a second fund. sol?

Simpler version for the recipe in $\mathbb{R}^2$ with double eigenvalue

1. Compute $\vec{v}$, eigenvector for r

2. Find $\vec{v}_2$ such that
$$(A - rI)^2 \vec{v}_2 = 0$$
note: This will usually be matrix 0.
and $\vec{v}_2$ not // to $\vec{v}$

3. Then compute $\vec{v}_1 = (A - rI)\vec{v}_2$

4. Then fund sol are
$$\vec{x}_1 = e^{rt}\vec{v}_1$$
$$\vec{x}_2 = te^{rt}\vec{v}_1 + e^{rt}\vec{v}_2$$

Back to our example

$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$ $\quad \lambda = 2$, double $\quad \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

1. eigenvector : $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

2. $(A-2I)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Thus any vector in $\mathbb{R}^2$ satisfies

$(A-2I)^2 \vec{u} = 0$

We choose ~~an~~ a simple vector, not // to $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{v}$

We choose $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

3 we compute

$$\vec{v}_1' = (A-2I)\vec{v}_2$$

$$= \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1^{st} \text{ column of } A-2I$$

$$= \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Note : the eigenvector was

$$\vec{v}' = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\vec{v}_1$$

4) We have found $r = 2$,

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Then

$$\vec{x}_1 = e^{rt}\vec{v}_1 = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\vec{x}_2 = t e^{rt}\vec{v}_1 + e^{rt}\vec{v}_2$$

$$= t e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

General sol. $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$

$$= e^{2t}\left(c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} e^{2t}\left((c_1+c_2) + c_2 t\right) \\ e^{2t}\left(-c_1 - c_2 t\right) \end{pmatrix}$$

Dominant direction

Dominant term as $t \to \infty$: $c_2 t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Example in $\mathbb{R}^3$

$$x' = Ax \quad \text{with } A = \begin{pmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{pmatrix}$$

Eigenvalues for A

$$\det(A - rI) = \begin{vmatrix} -r & 1 & 2 \\ -5 & -3-r & -7 \\ 1 & 0 & -r \end{vmatrix}$$

Expand wrt last row

$$= \begin{vmatrix} 1 & 2 \\ -3-r & -7 \end{vmatrix} - r \begin{vmatrix} -r & 1 \\ -5 & -3-r \end{vmatrix}$$

$$= -7 + 2(r+3) - r\left(r(r+3) + 5\right)$$

$$= -7 + 2r + 6 - r\left(r^2 + 3r + 5\right)$$

$$= -r^3 - 3r^2 - 3r - 1$$

-1 triple eigenvalue

$$= -\left(r^3 + 3r^2 + 3r + 1\right) = -(r+1)^3$$

$$A = \begin{pmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{pmatrix}$$

Eigenvectors for $\lambda = -1$

$$(A+I) = \begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} \boxed{1} & 0 & 1 \\ 1 & 1 & 2 \\ -5 & -2 & -7 \end{pmatrix} \underset{A_{13}(5)}{\overset{A_{12}(-1)}{\sim}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{pmatrix}$$

$$\overset{M_3(-\frac{1}{2})}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$u_3 = s$, free var.

Thus $u_3 = s \in \mathbb{R}$

$u_2 = -u_3 = -s$

$u_1 = -u_3 = -s$

$$\vec{u} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

We get 1 eigenvector for $s = -1$

Summary: we have $\lambda = -1$ triple eigenvalue, with 1 eigenvector
$\vec{u}' = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Then

1. Find generalized eigenvectors

we compute
$$(A+I)^3 = \begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix}^3$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
(check)

we take a simple $\vec{v}_3'$, not // to $\vec{u}'$. We chose
$$\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

True generalized eigenvectors

$$\vec{v}_2' = (A+I)\,\vec{v}_3'$$

$$= \begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix}$$

Then

$$\vec{v}_1' = \begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$$

$= -2\,\vec{u}'$, $\vec{u}'$ eigenvector

$\Rightarrow$ we keep $\vec{u}'$ instead of $\vec{v}_1$

2 Fund sol

$$\bar{x}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

$$\bar{x}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} e^{-t}$$

$$\bar{x}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \frac{t^2}{2} e^{-t} + \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-t}$$

Dominant term