Gallery of solutions (1)

Typo in Thm 8 Here A is 2×2

Case 2: 2 distinct $\mathbb{C}$-eigenvalues

Then $\lambda = \alpha \pm i\beta$,
$\vec{v} = \vec{a} + i\vec{b}$

Example of saddle point

$x' = A\, x \qquad A = \begin{pmatrix} 4 & 1 \\ 6 & -1 \end{pmatrix}$

Eigenvalues

$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 6 & -1-\lambda \end{vmatrix}$

$= (\lambda - 4)(\lambda + 1) - 6$

$= \lambda^2 - 3\lambda - 10$

Trivial $\qquad$ product $= -10$

Roots: $\lambda_1 = -2 \quad , \quad \lambda_2 = 5$

Eigenvector for λ_1: $(A + 2I)\vec{v} = 0$

$\Leftrightarrow \begin{pmatrix} 6 & 1 \\ \cancel{6} & \cancel{1} \end{pmatrix} \vec{v} = 0 \Leftrightarrow 6v_1 + v_2 = 0$

Take $\vec{v}_1 = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$

$$A = \begin{pmatrix} 4 & 1 \\ 6 & -1 \end{pmatrix}$$

③

Eigenvector for λ_2

$(A - 5I)\bar{v}' = 0$

$$\Longleftrightarrow \begin{pmatrix} -1 & 1 \\ \cancel{6} & \cancel{-6} \end{pmatrix} \bar{v}' = 0 \Longleftrightarrow v_2 = v_1$$

Take $\bar{v}_2' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

General solution

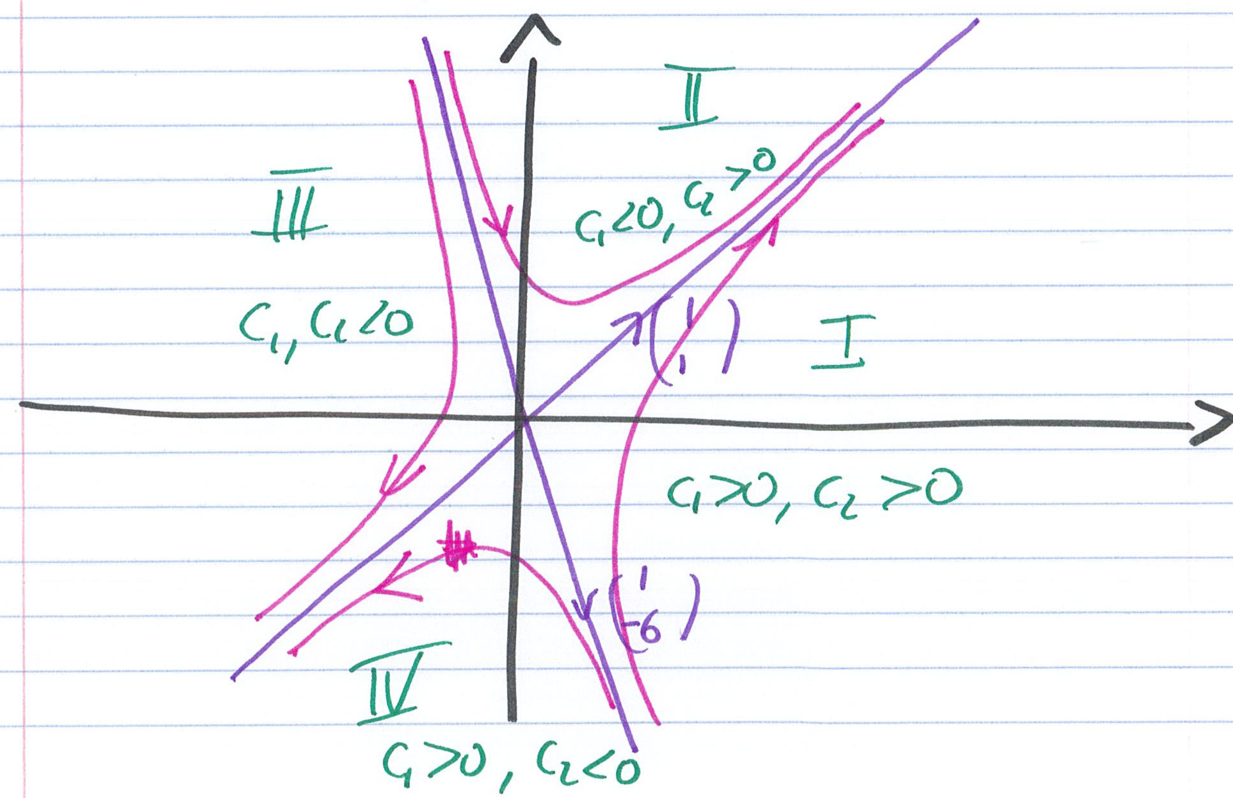
$$\vec{x}' = c_1 e^{-2t}\begin{pmatrix} 1 \\ -6 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Dominant term as $t \to -\infty$:

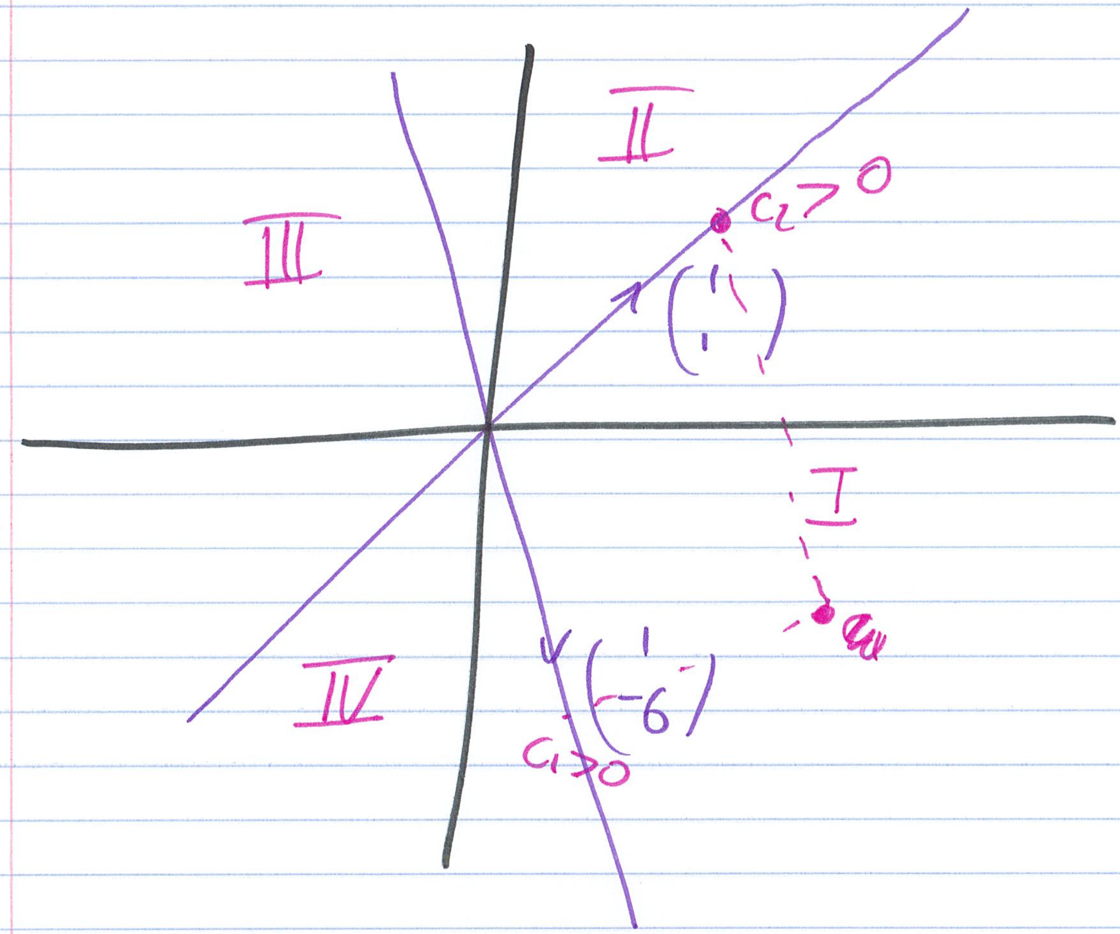
$$c_1 e^{-2t}\begin{pmatrix} 1 \\ -6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ -6 \end{pmatrix}$$ Dominant direction

Dominant term as $t \to +\infty$

$$c_2 e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ Dominant direction



5 + 6



Consider a vector

$$\vec{v} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -6 \end{pmatrix}$$

If $c_1, c_2 > 0 \rightarrow$ Quadrant I

$c_1 < 0,\ c_2 > 0 \rightarrow$ Quad. II

$c_1, c_2 < 0 \rightarrow$ Quad III

$c_1 > 0,\ c_2 < 0 \rightarrow$ Quad IV

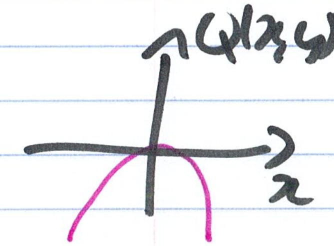
Why saddle point?

If we consider (recall $\lambda_1 = -2$, $\lambda_2 = 5$)

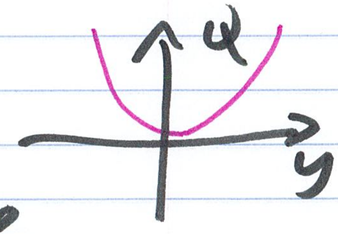
$$Q(x,y) = -2x^2 + 5y^2$$

Then at $(0,0) = (x,y)$ we have

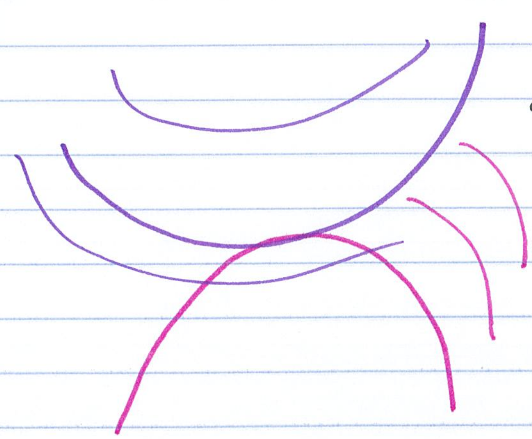
$x \mapsto Q(x,y)$ has max



$y \mapsto Q(x,y)$ has min



As a function of x,y we get



$\rightsquigarrow$ saddle shape

Example of sink : $\lambda_1 < \lambda_2 < 0$

$$x' = Ax \qquad A = \begin{pmatrix} -8 & 3 \\ 2 & -13 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -8-\lambda & 3 \\ 2 & -13-\lambda \end{vmatrix}$$

$$= (\lambda+8)(\lambda+13) - 6$$

$$= \lambda^2 + 21\lambda + 98$$

Quadratic formula: we get

(check)

$$\lambda_1 = -14 \qquad \lambda_2 = -7$$

Eigenvector for λ_1

$$(A + 14I)\vec{v} = 0 \iff \begin{pmatrix} \cancel{6} & \cancel{3} \\ 2 & 1 \end{pmatrix} \vec{v} = 0$$

$$\iff 2v_1 + v_2 = 0$$

Take $\vec{v}_1 = \begin{pmatrix} -1 \\ +2 \end{pmatrix}$

$$A = \begin{pmatrix} -8 & 3 \\ 2 & -13 \end{pmatrix}$$

Eigenvector for λ_2

$$(A+7I)\vec{v} = 0 \iff \begin{pmatrix} -1 & 3 \\ \cancel{2} & \cancel{-6} \end{pmatrix} \vec{v} = 0$$

$$\iff -v_1 + 3v_2 = 0$$

Take $\vec{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

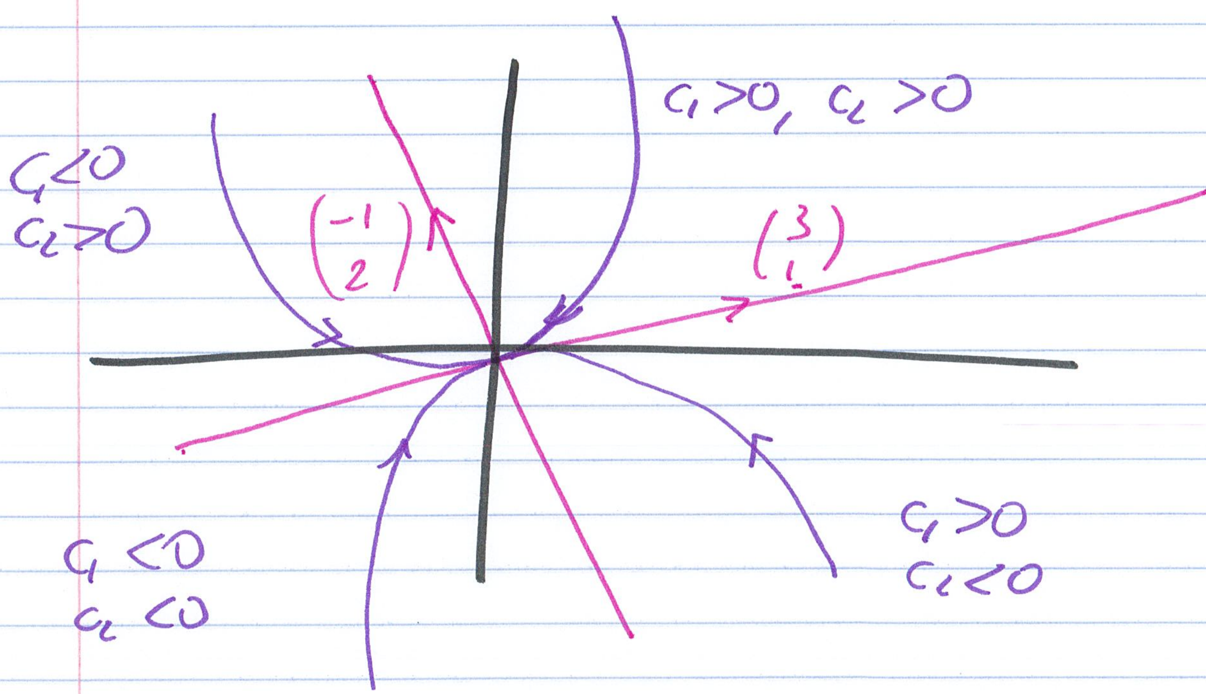
General solution

$$\vec{x}(t) = c_1 e^{-14t} \overbrace{\begin{pmatrix} -1 \\ 2 \end{pmatrix}}^{\vec{v}_1} + c_2 e^{-7t} \overbrace{\begin{pmatrix} 3 \\ 1 \end{pmatrix}}^{\vec{v}_2}$$

Dominant term as $t \to -\infty$

$$c_1 e^{-14t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \to \infty$$

Dominant term as $t \to +\infty$: $c_2 e^{-7t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \to 0$



Example of source

$x' = Bx$, $B = \begin{pmatrix} 8 & -3 \\ -2 & 13 \end{pmatrix} = -A$

Recall: eigenvalues for A were

$\lambda_1 = -14 \qquad \lambda_2 = -7$

Eigenvalues for $B = -A$:

If $A\vec{v}_1' = -14\vec{v}_1'$

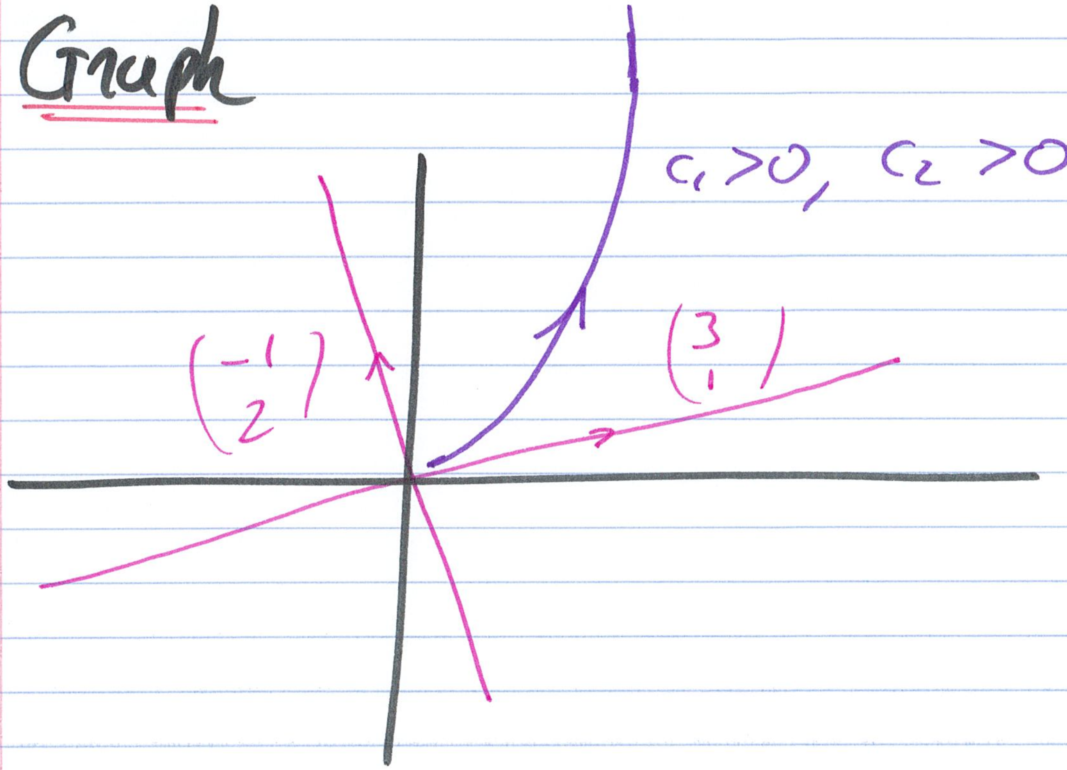
then $B\vec{v}_1' = -A\vec{v}_1' = 14\vec{v}_1'$

Thus 14 eigenvalue for B with eigenvector $\vec{v}_1'$

We get $\lambda_1 = 14 \quad \vec{v}_1' = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\lambda_2 = 7 \quad \vec{v}_2' = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Graph



$c_1 > 0, \ c_2 > 0$
$\begin{pmatrix} -1 \\ 2 \end{pmatrix}$
$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Line solution

$$x' = Ax \qquad A = \begin{pmatrix} -36 & -6 \\ 6 & 1 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -36-\lambda & -6 \\ 6 & 1-\lambda \end{vmatrix}$$

$$= (\lambda+36)(\lambda-1) + 36$$

$$= \lambda^2 + 35\lambda = \lambda(\lambda+35)$$

Thus $\lambda_1 = -35 \qquad \lambda_2 = 0$

$$v_1 = \begin{pmatrix} 6 \\ -1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$$

(check)

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General solution

$$\bar{x}'(t) = c_1 e^{-35t}\begin{pmatrix}6\\-1\end{pmatrix} + c_2\begin{pmatrix}1\\-6\end{pmatrix} \overset{1}{\cancel{e^{0t}}}$$

As $t \to -\infty$, $\bar{x}'(t) \to \infty$ along the $\begin{pmatrix}6\\-1\end{pmatrix}$ direction

As $t \to +\infty$, $\bar{x}'(t) \to c_2\begin{pmatrix}1\\-6\end{pmatrix}$