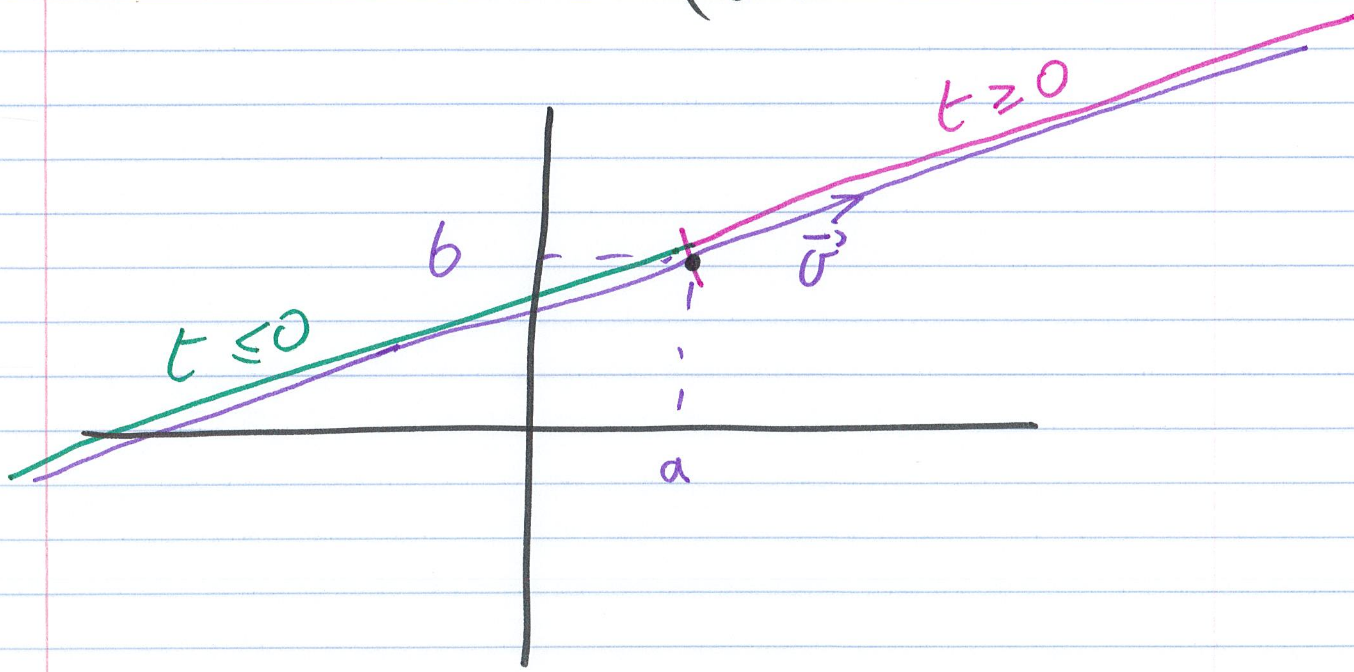
Gallery of curves (2)

Parametric eq for lines in $\mathbb{R}^2$

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + t \overbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}^{\vec{v}}$$

As $t \in \mathbb{R}$, we get the eq of a line with slope $\vec{v}$ and going through $\begin{pmatrix} a \\ b \end{pmatrix}$



Line solutions

$$x' = Ax \qquad A = \begin{pmatrix} -36 & -6 \\ 6 & 1 \end{pmatrix}$$

We have

$$\lambda_1 = -35 \quad \vec{v}_1 = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 0 \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

General solution

$$\vec{x} = c_1 \begin{pmatrix} 6 \\ -1 \end{pmatrix} e^{-35t} + c_2 \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -1 \end{pmatrix} \underbrace{c_1 e^{-35t}}_{r} + \underbrace{\begin{pmatrix} c_2 \\ 6c_2 \end{pmatrix}}_{\begin{pmatrix} a \\ b \end{pmatrix}}$$

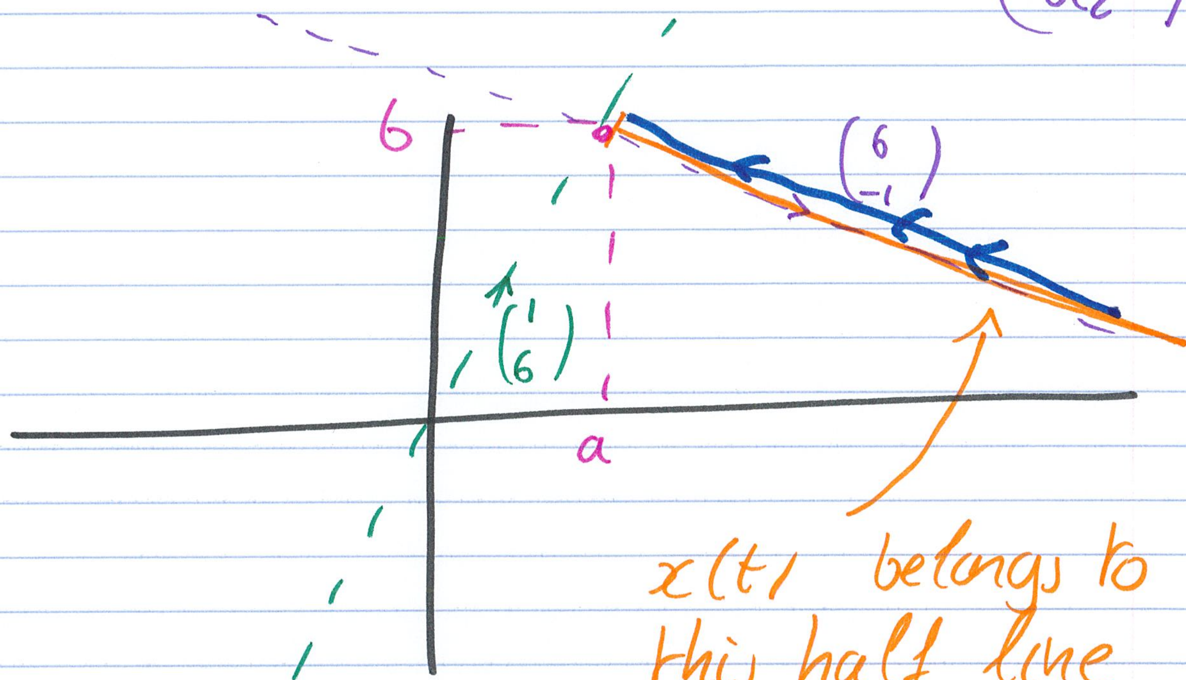
Rmk: If $c_1 > 0$, r takes all values in $\mathbb{R}_+$

If $c_1 < 0$, r takes all values in $\mathbb{R}_-$

Graph

$$\vec{x}(t) = \begin{pmatrix} 6 \\ -1 \end{pmatrix} \underbrace{\Omega}_{c_1 e^{-35t}} + \underbrace{\begin{pmatrix} a \\ 6 \end{pmatrix}}_{\begin{pmatrix} c_2 \\ 6c_2 \end{pmatrix}}$$

↳ Typo: should be $\begin{pmatrix} c_2 \\ -6c_2 \end{pmatrix}$



$x(t)$ belongs to this half line if $c_1 > 0$

Line solution (2)

$$x' = Bx \qquad B = \begin{pmatrix} 36 & 6 \\ -6 & -1 \end{pmatrix} = -A$$

Thus the eigenvalue decomposition for B is the same as for A, switching the sign of the eigenvalues we get

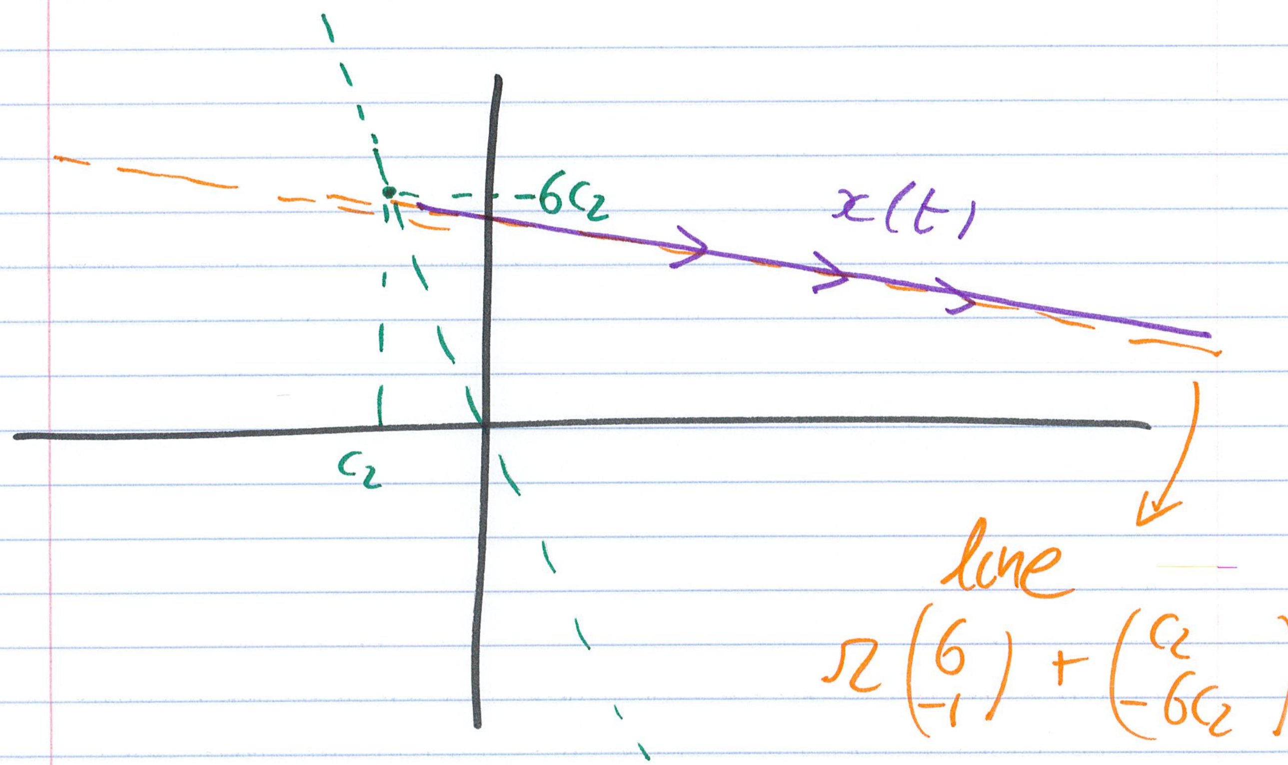
$$\lambda_1 = 35 \qquad v_1 = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 0 \qquad v_2 = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$$

General solution

$$\vec{x}(t) = c_1 e^{35t}\begin{pmatrix}6\\-1\end{pmatrix} + \begin{pmatrix}c_2\\-6c_2\end{pmatrix}$$

Assume $c_1 > 0$, $c_2 < 0$



Repeated eigen. with 2 eingenvect.

$x' = Ax \qquad A = 2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Then $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Thus $\lambda = 2$ ~~eig~~ double eigenvalue with $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

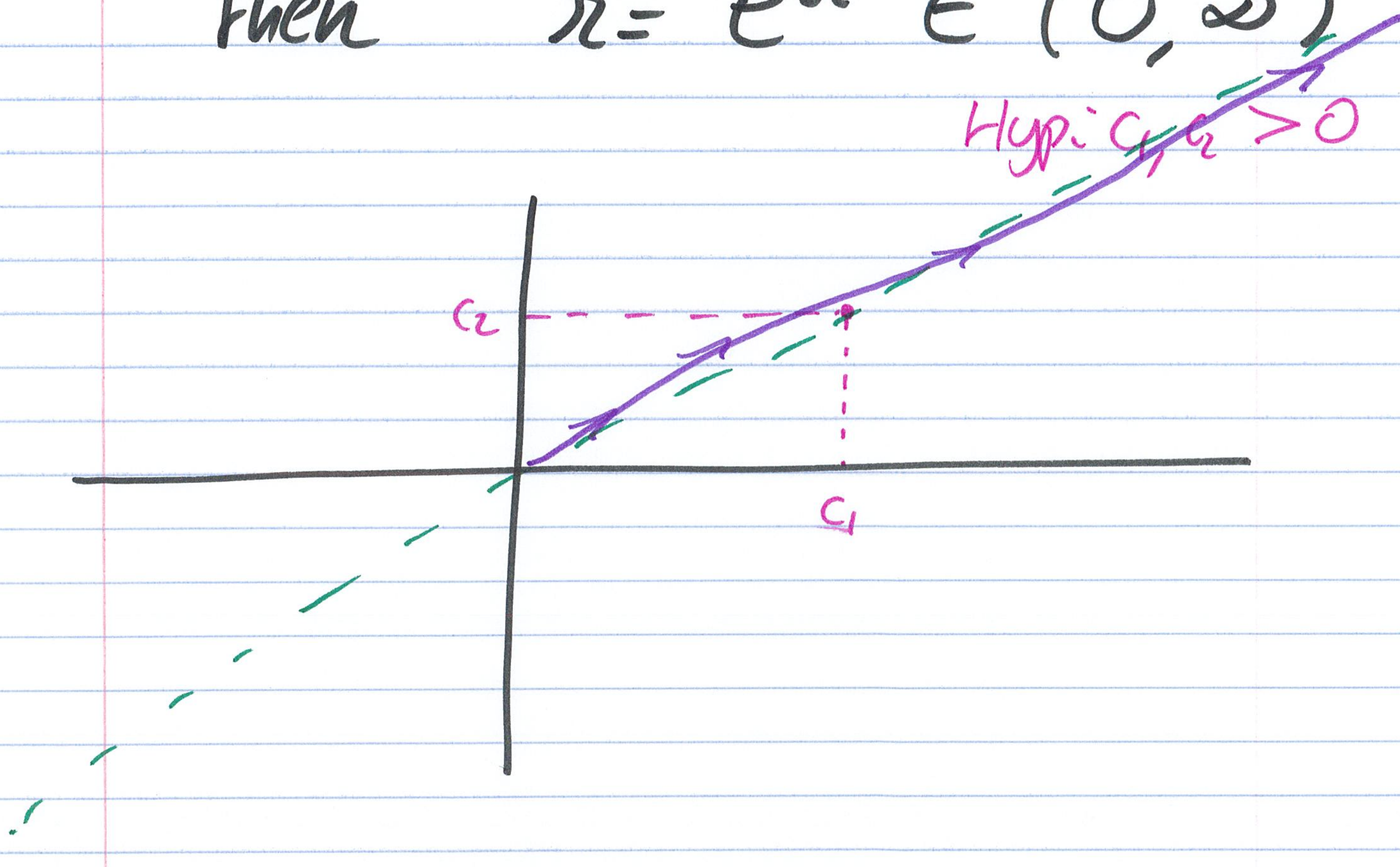
General solution

$$\bar{x}'(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$$

$$= e^{2t} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Note: If $t \in (-\infty, \infty)$

then $r = e^{2t} \in (0, \infty)$



Repeated eig. with 1 eigenvector

$$x' = \underbrace{\begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}}_{A} x$$

we get double eigenvalue $\lambda = 4$
with 1 eigenvector $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Generalized eigenvector:
Compute $(A - 4I)^2$

$$= \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}^2 = 0$$

Generalized eigenvector: $\vec{v}_2$ s.t.

$(A - 4I)^2 \vec{v}_2 = 0$ and $\vec{v}_2 \not\parallel \vec{v}_1$

we take $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Summary : we have $\lambda = 4$ rep.

$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

General solution :

$$\vec{x}(t) = c_1 e^{4t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{4t} \left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

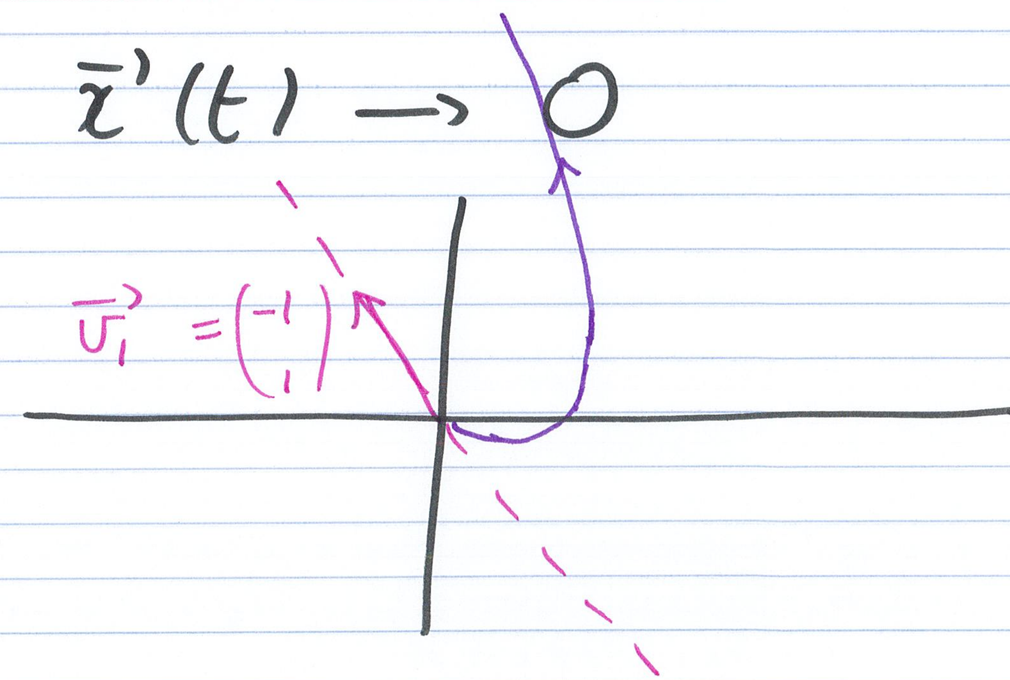
Dominant term as $t \to +\infty$

$$c_2 t e^{4t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

dominant direction

Limit as $t \to -\infty$

$$\vec{x}(t) \to 0$$



Repeated 0 eigen. , 1 eigenvector

$$x' = Ax, \qquad A = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} x$$

Check: 0 double eigenvalue with

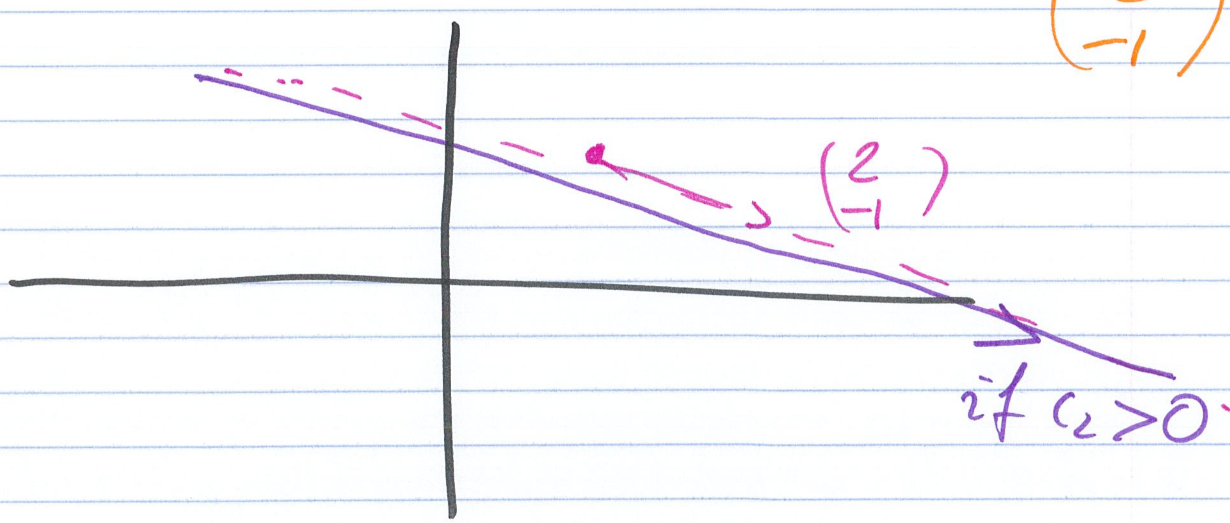
$$\vec{v}_1' = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \vec{v}_2' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

↳ generalized eigen vector

Gen sol.

$$\vec{x}'(t) = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} 2c_1 + c_2 \\ -c_1 \end{pmatrix} + \underbrace{c_2 t}_{r \in \mathbb{R}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \in \text{line with slope } \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$



Pure imag. eigenvalue

$x' = Ax \qquad A = \begin{pmatrix} 6 & -17 \\ 8 & -6 \end{pmatrix}$

Check: $\lambda_1 = 10i$

$\vec{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + i \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

Gen sol:

$\vec{x}(t) = c_1 \left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix} \cos(10t) - \begin{pmatrix} 5 \\ 0 \end{pmatrix} \sin(10t) \right\}$

$+ c_2 \left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix} \sin(10t) + \begin{pmatrix} 5 \\ 0 \end{pmatrix} \cos(10t) \right\}$

Init data: $x(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$. We get

$c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

We get $c_1 = c_2 = \frac{1}{2}$ (Check)

$\cos\left(\frac{\pi}{2}\right) = 0 \qquad \sin\left(\frac{\pi}{2}\right) = 1$

Unique solution

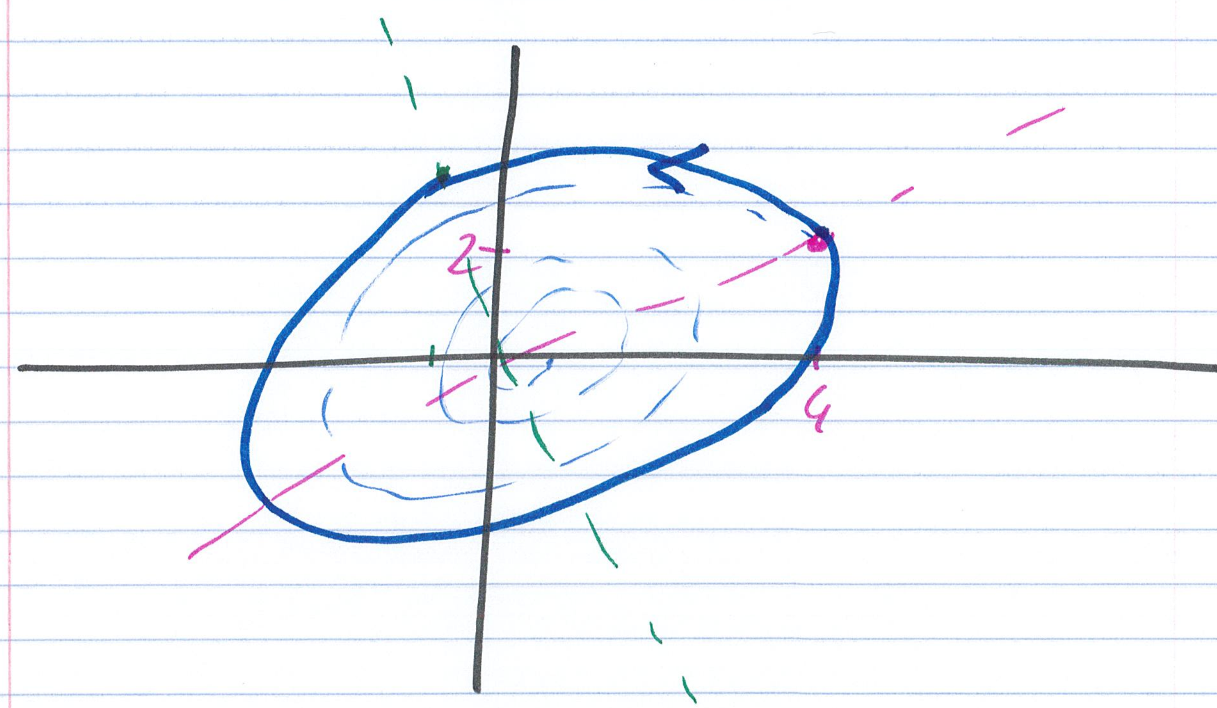
2 oscillating fct
↳ loops

$$\bar{x}'(t) = \begin{pmatrix} 4\cos(10t) - \sin(10t) \\ 2\cos(10t) + 2\sin(10t) \end{pmatrix}$$

Note: If $10t = 0$ we get

$$\bar{x}'(t) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

If $10t = \pi/2$, $\bar{x}'(t) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$



Eig. $\lambda = p + iq$ with $p < 0$

$$x' = Bx \qquad B = \begin{pmatrix} 5 & -17 \\ 8 & -7 \end{pmatrix} = A - Id$$

Then $\lambda_1 = -1 + 10i$

$$v_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + i \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

General sol: same as the previous one, multiplied be e^{-t}

With initial data

$$\vec{x}'(t) = e^{-t} \begin{pmatrix} 4\cos(10t) - \sin(10t) \\ 2\cos(10t) + 2\sin(10t) \end{pmatrix}$$

shrinks the ellipse