LESSON 4: SEPARABLE EQ

General separable equation

$$\int h(y)\,dy = \int g(x)\,dx$$

Rmk. • y is not an indep. variable
we still have $y = y(x)$

• In order to justify the integration on both sides, we chain rule

Equation $(1+y^2)\dfrac{dy}{dx} = x\cos(x)$

separable

$\Leftrightarrow (1+y^2)\,dy = x\cos(x)\,dx$

Integrate on both sides

lhs $\displaystyle\int (1+y^2)\,dy = y + \frac{y^3}{3}$ $(+c)$

rhs $\displaystyle\int \overset{u}{x}\,\overset{v'}{\cos(x)}\,dx$ $\quad u' = 1 \quad v = \sin(x)$

$\displaystyle = x\sin(x) - \int 1 \times \sin(x)\,dx$

$= x\sin(x) + \cos(x)$ $(+c)$

Solving the diff. equation

$$y + \frac{y^3}{3} = x\sin(x) + \cos(x) + c,$$

with $c \in \mathbb{R}$

implicit equation

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Equation $\quad x\,dx + y\,e^{-x}\,dy = 0$

$\Leftrightarrow \quad y\,e^{-x}\,dy = -x\,dx \qquad$ separable

$\times e^{x}$

$\Leftrightarrow \quad y\,dy = -x\,e^{x}\,dx$

Integrate on both sides

$$\int y\,dy = -\int \overset{u}{x}\,\overset{v'}{e^{x}}\,dx$$

$$\Leftrightarrow \frac{y^2}{2} = -x\,e^{x} + \int 1 \times e^{x}\,dx$$

$$\Leftrightarrow \frac{y^2}{2} = -x\,e^{x} + e^{x} + C_1 \,,\; C_1 \in \mathbb{R}$$

$$\Leftrightarrow \frac{y^2}{2} = (1-x)\,e^{x} + C_1$$

$$\Leftrightarrow y^2 = 2(1-x)\,e^{x} + C_2 \,,\; C_2 = 2C_1 \in \mathbb{R}$$

We have obtained the general solution

Gen sol: $y^2 = 2(1-x)e^x + C_2$

Initial condition: $x=0 \Rightarrow y=1$

Then

$$1^2 = 2(1-0)e^0 + C_2$$

$$\Rightarrow C_2 = 1-2 = -1$$

We get a unique solution:

$$y^2 = 2(1-x)e^x - 1$$

Here we can solve for y:

$$y = \pm\left(2(1-x)e^x - 1\right)^{\frac{1}{2}}$$

We determine $\pm$ sign thanks to the initial value ($x=0 \Rightarrow y=1$).

We get

$$1 = \pm\left(\overbrace{2(1-0)e^0 - 1}^{1}\right)^{\frac{1}{2}} = \pm 1$$

$\Rightarrow$ we choose $+$ sign. We get

$$y = \left(\underline{2(1-x)e^x - 1}\right)^{\frac{1}{2}}$$

Dom of def: when this is >0, $x \in (-1.7, 0.77)$

Application of Prop 3

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$$

$$\Leftrightarrow \quad \underbrace{2(y-1)}_{N(y)}\frac{dy}{dx} \underbrace{-(3x^2+4x+2)}_{M(x)} = 0$$

Then

$$H_1(x) = \int M(x)\,dx$$
$$= -\int (3x^2+4x+2)\,dx$$
$$= -(x^3+2x^2+2x)$$

$$H_2(y) = \int N(y)\,dy$$
$$= \int (2y-2)\,dy$$
$$= y^2-2y$$

General solution $\quad H_1(x)+H_2(y) = c$

$$y^2-2y-(x^3+2x^2+2x) = c$$

$$\Rightarrow \quad y^2-2y = x^3+2x^2+2x+c$$

$y^2 - 2y = x^3 + 2x^2 + 2x + c$ (Gen solution)

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Initial condition: $y(0) = -1$. We get

$(-1)^2 + 2 = c \Rightarrow c = 3$

Unique solution (implicit)

$y^2 - 2y = x^3 + 2x^2 + 2x + 3$

Solve for y: complete the square

$(y-1)^2 - 1 = x^3 + 2x^2 + 2x + 3$

$\Leftrightarrow (y-1)^2 = x^3 + 2x^2 + 2x + 4$

$\Leftrightarrow y = 1 \pm (x^3 + 2x^2 + 2x + 4)^{1/2}$

Determine $\pm$ sign: with $y(0) = -1$,

$-1 = 1 \pm 4^{1/2} = 1 \pm 2 \rightarrow -$ sign

Unique solution

$y = 1 - (x^3 + 2x^2 + 2x + 4)^{1/2}$

Rmk

- Domain of definition for y is $(-2, \infty)$
- With the general existence thm, we had that y was defined on $(-h, h)$ with a small h only

If $T > \tau$, then ~~th~~ temperature is decreasing

Cooling cup equation

$$\frac{dT}{dt} = -k(T-\tau), \quad T(0) = T_0$$

Hyp: $T_0 > \tau$ (hot coffee)

We have a separable equation:

$$\frac{dT}{T-\tau} = -k\, dt$$

Integrate on both sides:

$$\int \frac{dT}{T-\tau} = -k \int dt$$

$$\ln(|T-\tau|) = -kt + c_1 \qquad c_1 \in \mathbb{R}$$

Solve for T: exp on both sides

$$|T-\tau| = \overbrace{e^{c_1}}^{c_2 \in \mathbb{R}_+} e^{-kt}$$

$$T-\tau = \overbrace{\pm c_2}^{c_3 \in \mathbb{R}} e^{-kt}$$

$$T = \tau + c_3 e^{-kt}, \quad c_3 \in \mathbb{R}$$

Gen sol: $T = \tau + c_3 e^{-kt}$

<u>Initial condition</u> $t=0 \Rightarrow T = T_0$

We get

$$T_0 = \tau + c_3 e^0 \Leftrightarrow c_3 = T_0 - \tau$$

Unique solution (hot coffee)

$$T(t) = T = \tau + \overbrace{(T_0 - \tau)}^{>0} e^{-kt}$$

<u>Question</u>

$$\lim_{t\to\infty} T(t) =$$

$$= \lim_{t\to\infty} \tau + (T_0 - \tau)\overbrace{e^{-kt}}^{\to 0 \text{ as } t\to\infty} = \tau$$

<u>Rmk</u> . We could have obtained the limiting behavior with the slope field

. Additional information: exponential decay of T to τ.