Outline

General form of 1st order linear equation

General 1^{st} order diff eq : $y' = f(t,y)$

Linear eq: when f is linear in y

General form 1:

$$\frac{dy}{dt} + p(t)y = g(t)$$

↳ linear in y

General form 2:

$$P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

Remark:

2 forms are equivalent if $P(t) \neq 0$

From form 2 to form 1

$$P(t)\,y' + Q(t)\,y = G(t)$$

$$\overset{\times \frac{1}{P}}{\iff} \quad y' + \underbrace{\frac{Q(t)}{P(t)}}_{p(t)}\, y = \underbrace{\frac{G(t)}{P(t)}}_{g(t)} \longrightarrow \text{form 1}$$

Rmk This can be done as long as $P(t) \neq 0$

When $P(t) = 0$, we get singularity issues

Example with direct integration

Equation:

$$\left(4+t^2\right)\frac{dy}{dt}+2t\,y=4t$$

Equivalent form:

$$\frac{d}{dt}\left[\left(4+t^2\right)y\right]=4t$$

General solution: For a constant $c \in \mathbb{R}$,

$$y=\frac{2t^2+c}{4+t^2}$$

Equation $\quad \underbrace{(4+t^2)}_{f(t)}\underbrace{y'}_{g'(t)} + \underbrace{2t}_{f'(t)}\underbrace{y}_{g(t)} = 4t$

Note : $f'(t) = (4+t^2)' = 2t$

Recall $\quad (fg)' = f'g + fg'$

Thus the equation can be written as

$$(fg)' = 4t$$

$$\Leftrightarrow \quad \left((4+t^2)y\right)' = 4t$$

Integrate on both sides :

$$(4+t^2)y = \int (4t)dt = 2t^2 + c$$

General solution

$$(4+t^2)y = \int (4t)\,dt = 2t^2 + c$$

$$\Leftrightarrow \quad y = \frac{2t^2+c}{4+t^2}, \quad \text{for } c \in \mathbb{R}$$

Q1 Will this simple trick work for every linear eq?

↳ No (this relied on $2t = (4+t^2)'$)

Q2 Can we generalize the trick to almost any linear eq?

↳ yes (integrating factor)

Method of integrating factor

has to be fun !

General equation:

$$\frac{dy}{dt} + p(t)y = g(t) \tag{10}$$

Recipe for the method:

1. Consider equation (10)
2. Multiply the equation by a function μ mu
3. Try to choose μ such that equation (10) is reduced to:

$$\frac{d(\mu y)}{dt} = a(t) \tag{11}$$

4. Integrate directly equation (11)

Notation: If previous recipe works, μ is called integrating factor

Example of integrating factor (1)

Equation:

$$\frac{dy}{dt} + \frac{1}{2}\,y = \frac{1}{2}\,e^{t/3} \qquad (12)$$

Equation $\left(y' + \frac{1}{2} y = \frac{1}{2} e^{t/3}\right)$ $\times \mu$

① The linear eq is already under form 1

② $\mu y' + \frac{1}{2}\mu\, y = \frac{1}{2} e^{t/3} \mu$

we wish this term to be μ'

In this way the equation becomes

$$\mu y' + \mu' y = \frac{1}{2} e^{t/3} \mu$$ (back to simple example)

③ Find μ s.t. $\mu' = \frac{1}{2}\mu$

(i) $\mu = \frac{1}{4} t^2 \Rightarrow \mu' = \frac{2t}{4} = \frac{t}{2}$ $\longrightarrow$ NO

(ii) $\mu = e^{t/2} \Rightarrow \mu' = \frac{1}{2} e^{t/2} = \frac{1}{2}\mu$

We choose $\mu = e^{t/2}$

③ - Ctd We have obtained

$$e^{t/2} y' + \frac{1}{2} e^{t/2} y = \frac{1}{2} \overbrace{e^{t/3} e^{t/2}}^{= e^{5t/6}}$$

$$\Leftrightarrow \quad (e^{t/2} y)' = \frac{1}{2} e^{5t/6}$$

④ Integrate on both sides

$$e^{t/2} y = \frac{1}{2} \int e^{\frac{5t}{6}} dt$$

$$\Leftrightarrow \quad e^{t/2} y = \frac{1}{2} \times \frac{6}{5} e^{\frac{5t}{6}} + c$$

$$\Leftrightarrow \quad e^{t/2} y = \frac{3}{5} e^{5t/6} + c$$

$\times e^{-t/2}$

$$\Leftrightarrow \quad y = \frac{3}{5} e^{5t/6 - t/2} + c\, e^{-t/2}$$

$$\Leftrightarrow \quad y = \frac{3}{5} e^{t/3} + c\, e^{-t/2}$$

General solution

$$y = \frac{3}{5} e^{t/3} + c\, e^{-t/2}$$

Question : what is the dominant term as $t \to \infty$

We have $\quad e^{-t/2} \longrightarrow 0$ as $t \to \infty$

$\qquad\qquad\; e^{t/3} \longrightarrow \infty$ as $t \to \infty$

Thus $\boxed{y \simeq \frac{3}{5} e^{t/3}}$ as $t \to \infty$

Example of integrating factor (2)

Multiplication by μ:

$$\mu(t)\frac{dy}{dt} + \frac{1}{2}\,\mu(t)\,y = \frac{1}{2}\,\mu(t)\,e^{t/3}$$

Integrating factor: Choose μ such that $\mu' = \frac{1}{2}\,\mu$, i.e $\mu(t) = e^{t/2}$

Solving the equation: We have, for $c \in \mathbb{R}$

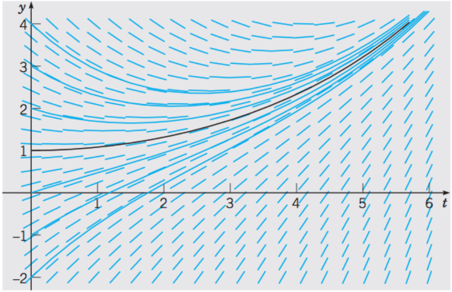
$$\begin{aligned}(12) \quad &\Longleftrightarrow \quad \frac{d\left(e^{t/2}y\right)}{dt} = \frac{1}{2}\,e^{\frac{5t}{6}} \\ &\Longleftrightarrow \quad y(t) = \frac{3}{5}\,e^{\frac{t}{3}} + c\,e^{-\frac{t}{2}}\end{aligned}$$

Example of integrating factor (3)

Solution for a given initial data: If we know $y(0) = 1$, then

$$y(t) = \frac{3}{5}\, e^{\frac{t}{3}} + \frac{2}{5}\, e^{-\frac{t}{2}}$$

Direction fields and integral curves:



How did we find μ ?

μ solves $\quad \mu' = \frac{1}{2}\mu$

$$\Leftrightarrow \quad \frac{d\mu}{\mu} = \frac{1}{2}\, dt \quad \text{(separable)}$$

$$\Leftrightarrow \quad \ln \mu = \frac{1}{2} t$$

$$\Leftrightarrow \quad \mu = e^{\frac{1}{2}t}$$

General case with constant coefficient

Proposition 4.

Equation considered:

$$\frac{dy}{dt} + ay = g(t), \quad \text{and} \quad y(t_0) = y_0. \tag{13}$$

Main idea: $\mu = e^{at}$

Hypothesis:

$$a \in \mathbb{R}, \qquad g : \mathbb{R}_+ \longrightarrow \mathbb{R} \text{ continuous.}$$

Then general solution to (13) is given by:

$$y(t) = e^{-at} \int_{t_0}^{t} e^{as} \, g(s) \, ds + c \, e^{-at}.$$

with $t_0 \geq 0$ and $c \in \mathbb{R}$.

Example with exponential growth

Equation:

$$\frac{dy}{dt} - 2y = 4 - t$$

$$a = -2$$

$$\Rightarrow \mu = e^{-2t}$$

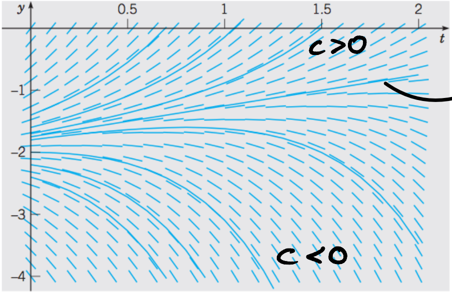
Example with exponential growth (2)

General solution: for $c \in \mathbb{R}$,

leading term

$$y(t) = -\frac{7}{4} + \frac{t}{2} + c\, e^{2t}$$

Direction fields and integral curves:



c=0 (linear function)

General first order linear case

Proposition 5.

Equation considered:

$$\frac{dy}{dt} + p(t)y = g(t), \tag{14}$$

solution of $\mu' = \mu p$

Integrating factor:

$$\boxed{\mu(t) = \exp\left(\int p(r)\, dr\right).}$$

Then general solution to (14) is given by:

$$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)\, g(s)\, ds + c\right].$$

with $t_0 \geq 0$ and $c \in \mathbb{R}$.

Example with unbounded p (1)

Equation considered:

$$t\,y' + 2y = 4t^2, \qquad y(1) = 2. \tag{15}$$

Equation $\quad t y' + 2y = 4t^2 \quad$ (fum 2)

① Eq $\Leftrightarrow \quad y' + \frac{2}{t} y = 4t$

② Integrating factor:

$$\mu(t) = \exp\left(\int \frac{2}{t} \, dt \right)$$

$$= \exp\left(2 \ln(t) \; (+c) \right) = \exp(\ln(t^2))$$

$$= t^2$$

③ Eq $\Leftrightarrow \quad (t^2 y)' = 4t \times t^2 = 4t^3$

④ Integrate on both sides

$$t^2 y = \int 4t^3 \, dt = t^4 + c$$

General solution Divide by t^2. We get

$$y = \frac{t^4}{t^2} + \frac{c}{t^2}$$

$$y = t^2 + \frac{c}{t^2}$$

Initial condition $y(1) = 2$. We get

$$2 = 1^2 + \frac{c}{1^2} \implies c = 1$$

Unique solution: Rmk: Here we get a singular solution at $t=0$

$$y = t^2 + \frac{1}{t^2}$$

This is due to original eq

$$y' = \frac{2}{t} y + 4t$$

singular

Example with unbounded p (2)

Equivalent form:

$$y' + \frac{2}{t} y = 4t, \qquad y(1) = 2.$$

Integrating factor:

$$\mu(t) = t^2.$$

Solution:

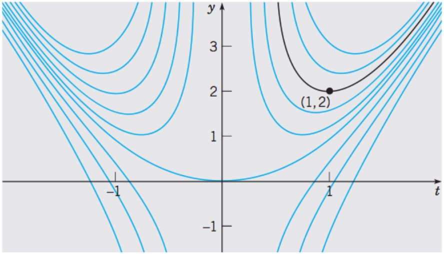
$$y(t) = t^2 + \frac{1}{t^2} \tag{16}$$

Example with unbounded p (2)

Gen sol:
$y = t^2 + \frac{c}{t^2}$

If $c=0$,
$y = t^2$
(not singular)

Some integral curves:



Comments:

1. Example of solution which is not defined for all $t \geq 0$
2. Due to singularity of $t \mapsto \frac{1}{t}$
3. Integral curves for $t < 0$: not part of initial value problem
4. According to value of $y(1)$, different asymptotics as $t \to 0$
5. Boundary between 2 behaviors: function $y(t) = t^2$

Example with no analytic solution (1)

Equation considered:

$$2y' + t\,y = 2.$$

Equation $\quad 2y' + ty = 2$

① Form 1: $\left(y' + \frac{t}{2} y = 1\right) \times e^{t^2/4}$

② Integrating factor: $\mu = \exp\left(\int \frac{t}{2}\, dt\right)$

$$= \exp\left(\frac{t^2}{4}\right)$$

③ Equation becomes

$$\left(e^{t^2/4} y\right)' = e^{t^2/4}$$

No known expression for $\int e^{\text{quadratic}}$

④ Integrate on both sides

$$e^{t^2/4} y = \int e^{t^2/4}\, dt$$

General solution

$$y = e^{-t^2/4}\left(\int_0^t e^{s^2/4}\, ds + c\right)$$