# Outline

1. Population models
2. Equilibrium solutions and stability
3. Numerical approximation: Euler's method

# Malthusian growth

Hypothesis:
Rate of change proportional to value of population

Equation: for $k \in \mathbb{R}$ and $P_0 \geq 0$,

$$\frac{dP}{dt} = k\, P, \qquad P(0) = P_0$$

Solution:

$$P = P_0 \exp(kt)$$

# Solving Malthusian growth

Eq
$$\frac{dP}{dt} = k P$$
($P \geq 0$ since we have a population)

$$\iff \frac{dP}{P} = k \, dt$$

Integrate on both sides

$$\ln P = k t + c_1$$

$\exp(\cdot)$

$$\iff P = c_2 e^{kt}$$

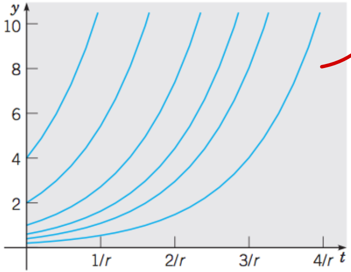
With initial condition $P(0) = P_0$, we get

$$\boxed{P(t) = P_0 e^{kt}}$$

# Exponential growth (2)

Not a sustainable growth

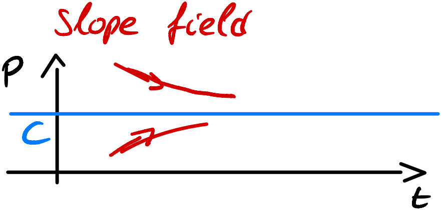
Integral curves:




Limitation of model:

- Cannot be valid for large time $t$.

# Logistic population model




**Basic idea:**

- Growth rate decreases when population increases.

**Model:**

$$\frac{dP}{dt} = r \underbrace{\left(1 - \frac{P}{C}\right)}_{\text{extra term w.r.t Malthus}} P, \tag{1}$$

This term is $>0$ if $P<C$

It is $<0$ if $P>C$

where

- $r \equiv$ reproduction rate
- $C \equiv$ carrying capacity

## Information from slope field

(i) If $P_0 < C$, then

$$\lim_{t \to \infty} P(t) = C$$

and $t \mapsto P(t)$ is $\nearrow$

(ii) If $P_0 > C$, then

$$\lim_{t \to \infty} P(t) = C$$

and $t \mapsto P(t)$ is $\searrow$

## Solving the logistic eq

$$\frac{dP}{dt} = r\left(1 - \frac{P}{C}\right)P$$

$$\Leftrightarrow \quad \frac{dP}{\left(1 - \frac{P}{C}\right)P} = r\,dt \quad \longrightarrow \text{separable}$$

$$\Leftrightarrow \quad \frac{C}{(C-P)P}\,dP = r\,dt$$

Integrating the lhs

$$\frac{c}{(c-P)P} = f(P) = \frac{a}{c-P} + \frac{b}{P}$$

numbers to be computed

In order to compute $a$, let us write

$$(c-P)f(P)\big|_{c=P} \overset{\text{lhs}}{=} \frac{c}{P}\Big|_{c=P} = 1$$

$$(c-P)f(P)\big|_{P=c} \overset{\text{rhs}}{=} a + b\,\frac{(c-P)}{P}\Big|_{P=c} = a$$

Thus $\boxed{a=1}$

Compute $b$

$$Pf(P)\big|_{P=0} \overset{\text{lhs}}{=} \frac{c}{c-0} = 1$$

$$Pf(P)\big|_{P=0} \overset{\text{rhs}}{=} b$$

Thus $\boxed{b=1}$

Integrating lhs - Ctd  we have obtained

$$f(P) = \frac{1}{C-P} + \frac{1}{P}$$

Integrate

$$\int f(P)\,dP = \int \left( \frac{1}{C-P} + \frac{1}{P} \right) dP$$

$$= - \ln(|C-P|) + \ln(|P|) \quad (+c)$$

$$= \ln\left( \left| \frac{P}{C-P} \right| \right)$$

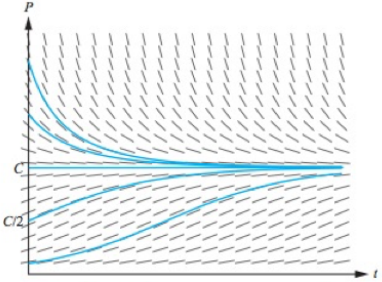
Integrating rhs  $r \int dt = rt \quad (+c)$

Eq for $P$  $$\ln\left( \left| \frac{P}{C-P} \right| \right) = rt + c_1$$

# Logistic model: qualitative study

Information from slope field:

- Equilibrium at $P = C$
- If $P < C$ then $t \mapsto P$ increasing
- If $P > C$ then $t \mapsto P$ decreasing
- Possibility of convexity analysis



# Logistic model: solution

First observation: Equation (1) is separable

Integration: Integrating on both sides of (1) we get

$$\ln\left(\left|\frac{P}{C-P}\right|\right) = rt + c_1$$

which can be solved as:

$$P(t) = \frac{c_2 C}{c_2 + e^{-rt}}$$

Initial value problem: If $P_0$ is given we obtain

$$P(t) = \frac{C\,P_0}{P_0 + (C - P_0)e^{-rt}}$$

# Information obtained from the resolution

Asymptotic behavior:

$$\lim_{t \to \infty} P(t) = C$$

Prediction: If

- Logistic model is accurate
- $P_0$, $r$ and $C$ are known

Then we know the value of $P$ at any time $t$